Derived Categories in Algebraic Geometry

Andreas Krug

Habilitation Thesis

3rd of April 2017
# Contents

0 Introduction .............................................. 7
  0.1 Background, history, and state of the art .................. 8
    0.1.1 Derived functors and relative Duality .................. 8
    0.1.2 Fourier-Mukai partners \( \mathbf{P} \) .......................... 10
    0.1.3 Autoequivalences \( \mathbf{P} \) ............................. 12
    0.1.4 Semi-orthogonal decompositions and exceptional sequences \( \mathbf{P} \) 16
    0.1.5 Birational geometry and derived categories ............... 18
    0.1.6 McKay correspondence ................................ 18
    0.1.7 Categorification \( \mathbf{P} \) ............................. 20
    0.1.8 Derived categories of Hilbert schemes of points on surfaces 20
  0.2 Summary of results .................................. 23
    0.2.1 Equivariant derived functors and a necessary condition for derived McKay equivalences [1] 23
    0.2.2 McKay correspondence for cyclic quotients and categorical crepant resolutions [2] ...................... 24
    0.2.3 Spherical functors on the Kummer surface [3] .......... 25
    0.2.4 Varieties with \( \mathbb{P} \)-units [4] .......................... 26
    0.2.5 Semi-orthogonal decompositions on Hilbert schemes of points on surfaces and induced autoequivalences [5] 27
    0.2.6 \( \mathbb{P} \)-functor versions of the Nakajima operators [6; 7] 28
    0.2.7 Categorification of the Heisenberg action [8] ........... 31
    0.2.8 Tautological bundles and the McKay correspondence in the reverse direction [9] ...................... 31

1 Equivalences of equivariant derived categories ................. 39
  1.1 Introduction ....................................... 39
  1.2 Preliminaries ...................................... 40
    1.2.1 Group actions on categories ......................... 40
    1.2.2 Smooth projective stacks and Fourier-Mukai transforms 42
  1.3 Lifts of equivalences to equivariant categories .......... 43
    1.3.1 Linearisations by standard autoequivalences .......... 43
    1.3.2 Fourier-Mukai transforms of \( \mu \)-type ............... 44
    1.3.3 Equivariance of functors vs. equivariance of kernels 44
    1.3.4 Equivariant lifts .................................. 45
    1.3.5 Monoidal and \( \chi \)-linear autoequivalences ........... 46
    1.3.6 Lifts of adjunctions .............................. 47
1.3.7 Lifts of spherical and $\mathbb{P}^n$-functors .................................. 48
1.4 Geometric Interpretation ............................................................. 51
1.5 Applications .................................................................................. 52
  1.5.1 Hochschild homology ................................................................. 52
  1.5.2 Galois coverings induced by characters ....................................... 54
  1.5.3 Stacks with characters as canonical bundles .................................. 55
  1.5.4 Symmetric quotients and generalised Kummer stacks .................. 56
  1.5.5 Kummer stacks ......................................................................... 57
  1.5.6 Enriques quotients of Hilbert schemes ........................................ 57
  1.5.7 Calabi-Yau covers of Hilbert schemes ......................................... 59
1.6 Appendix: Necessary condition for lifts ........................................... 59

2 Derived categories of resolutions of cyclic quotient singularities 63
  2.1 Introduction ................................................................................. 63
  2.2 Preliminaries ................................................................................ 66
    2.2.1 Fourier-Mukai transforms and kernels ..................................... 66
    2.2.2 Group actions and derived categories ..................................... 67
    2.2.3 Semi-orthogonal decompositions ............................................ 67
    2.2.4 Dual semi-orthogonal decompositions .................................... 68
    2.2.5 Linear functors and linear semi-orthogonal decompositions ....... 69
    2.2.6 Relative Fourier-Mukai transforms ....................................... 70
    2.2.7 Relative tilting bundles .......................................................... 71
    2.2.8 Spherical functors ................................................................. 74
  2.3 The geometric setup ..................................................................... 74
    2.3.1 The resolution as a moduli space of $G$-clusters ....................... 75
  2.4 Proof of the main result ............................................................... 78
    2.4.1 Generators and linearity ......................................................... 79
    2.4.2 On the equivariant blow-up .................................................... 80
    2.4.3 On the cyclic cover ............................................................... 81
    2.4.4 The case $m \geq n$ ............................................................... 82
    2.4.5 The case $n \geq m$ ............................................................... 85
    2.4.6 The case $m = n$: spherical twists and induced tensor products ... 86
  2.5 Categorical resolutions ................................................................. 89
    2.5.1 General definitions ............................................................... 89
    2.5.2 The weakly crepant neighbourhood in the cyclic setup ............. 90
    2.5.3 The discrepant category and some speculation ......................... 91
    2.5.4 (Non-)unicity of categorical crepant resolutions ..................... 92
    2.5.5 Connection to Calabi-Yau neighbourhoods ............................ 93
  2.6 Stability conditions for Kummer threefolds ................................... 95

3 Spherical functors on the Kummer surface 99
  3.1 Introduction ................................................................................. 99
  3.2 Natural Functors on the Kummer Surface ..................................... 101
4 Varieties with $\mathbb{P}$-units

4.1 Introduction ........................................... 106
4.2 Notations and preliminaries ............................. 108
  4.2.1 Notations and conventions ............................ 108
  4.2.2 Derived categories of coherent sheaves ............ 109
  4.2.3 Special objects of the derived category .......... 110
4.3 Hyperkähler and Enriques varieties ....................... 111
  4.3.1 Hyperkähler manifolds ............................. 111
  4.3.2 Automorphisms and their action on cohomology ... 112
  4.3.3 Proof of Observation 4.1.2 ......................... 113
  4.3.4 Enriques varieties .................................. 115
  4.3.5 Enriques stacks .................................... 117
4.4 Construction of varieties with $\mathbb{P}^n[k]$-units .... 118
  4.4.1 Definition and basic properties ....................... 118
  4.4.2 Non-examples ...................................... 118
  4.4.3 Main construction method ............................ 119
4.5 Structure of varieties with $\mathbb{P}^n[k]$-units .......... 120
  4.5.1 General properties .................................. 120
  4.5.2 The case $k = 4$ .................................... 120
4.6 Further remarks .......................................... 124
  4.6.1 Further constructions using strict Enriques varieties . 124
  4.6.2 A construction not involving strict Enriques varieties . 125
  4.6.3 Possible construction for $k = 6$ .................... 126
  4.6.4 Stacks with $\mathbb{P}^n[k]$-units ................... 126
  4.6.5 Derived invariance of strict Enriques varieties ..... 126
  4.6.6 Autoequivalences of varieties with $\mathbb{P}^n[k]$-unit 127
  4.6.7 Varieties with $\mathbb{P}^n[k]$-units as moduli spaces 129

5 On the derived category of the Hilbert scheme of points on an Enriques surface .... 131

5.1 Introduction ........................................... 131
5.2 Preliminaries ........................................... 133
  5.2.1 Hilbert schemes of surfaces with $p_g = q = 0$ .... 133
  5.2.2 Canonical covers .................................... 133
  5.2.3 Fourier–Mukai transforms and kernels ............. 133
  5.2.4 Equivalences of canonical covers .................... 134
  5.2.5 Spherical functors .................................. 134
  5.2.6 $\mathbb{P}^n$-functors ................................ 134
  5.2.7 Semi-orthogonal decompositions ...................... 135
  5.2.8 Group actions and derived categories .............. 136
5.3 Proofs of the main results ................................ 136
  5.3.1 Surfaces with $p_g = q = 0$ ........................ 136
  5.3.2 Application to Enriques surfaces .................... 137
  5.3.3 Comparison to known autoequivalences .............. 139
5.4 Exceptional sequences on $X^{[n]}$ ........................ 142
5.5 The truncated universal ideal functor .................... 144
  5.5.1 The case of an even dimensional Calabi–Yau variety 146
5.5.2 The case $H^*(O_Z) = \mathbb{C}[0]$ ........................................ 147
5.5.3 The orthogonal complement of $\text{im } G$ .......................... 147

6 On derived autoequivalences of Hilbert schemes and generalized Kummer varieties 151
6.1 Introduction ......................................................... 151
6.2 $\mathbb{P}^n$-functors .................................................. 153
6.3 The diagonal embedding ............................................ 156
6.4 Composition with the Bridgeland–King–Reid–Haiman equivalence 159
6.5 Comparison with other autoequivalences ............................ 162
6.6 $\mathbb{P}^n$-objects on generalised Kummer varieties 165

7 $\mathbb{P}$-functor versions of the Nakajima operators 169
7.1 Introduction .......................................................... 169
7.1.1 Main results ...................................................... 170
7.1.2 Structure of the proof ........................................... 172
7.1.3 Similarities to the Nakajima operators .......................... 173
7.1.4 The induced autoequivalences ................................ 174
7.2 Definition of the functors .......................................... 175
7.2.1 Equivariant Fourier–Mukai transforms ........................ 175
7.2.2 $\mathbb{P}$-functors .................................................. 176
7.2.3 Notations and conventions ...................................... 176
7.2.4 The Fourier–Mukai kernel ...................................... 177
7.2.5 Adjoint kernels .................................................. 177
7.2.6 Description of the functor ...................................... 178
7.3 Techniques and examples .......................................... 179
7.3.1 Derived intersections .......................................... 179
7.3.2 Partial diagonals and the standard representation .......... 180
7.3.3 The case $\ell = 0$ .................................................. 182
7.3.4 The approach for general $\ell$ .................................. 182
7.3.5 Invariants of inflations ........................................ 182
7.3.6 The case $\ell = 1$ .................................................. 183
7.3.7 Orthogonality in the curve case ................................. 185
7.3.8 Non-orthogonality in the surface case ......................... 185
7.3.9 The case $\ell = 2, n = 2$ ........................................ 185
7.4 Proof of the main results ........................................... 186
7.4.1 Computation of the direct summands .......................... 186
7.4.2 The induced maps ............................................... 187
7.4.3 Computation of the $\mathcal{P}_{iR} \star \mathcal{P}_j$ ....................... 188
7.4.4 Spectral sequences .............................................. 189
7.4.5 Long exact sequences ......................................... 189
7.4.6 The curve case: induced maps ................................ 190
7.4.7 The curve case: fully faithfulness ............................ 192
7.4.8 The curve case: orthogonality ................................ 192
7.4.9 The surface case: induced maps .............................. 193
7.4.10 The surface case: cohomology ............................... 195
7.4.11 The surface case: splitting and monad structure ......... 197
7.4.12 The case of the generalised Kummer stacks ........................................ 198
7.5 Interpretation of the results ................................................................. 201
  7.5.1 Spherical and \( \mathbb{P} \)-twists .................................................. 201
  7.5.2 The case \( n = 1 \) and comparison to [CL12] .................................. 201
  7.5.3 Induced autoequivalences on the Hilbert schemes ............................ 203
  7.5.4 Induced autoequivalences on the Kummer varieties .......................... 205
  7.5.5 Relation to the (truncated) universal ideal functors ......................... 206
  7.5.6 Braids on hyperkähler fourfolds ............................................... 207
  7.5.7 Semi-orthogonal decompositions in the curve cases ......................... 208
  7.5.8 Induced autoequivalences in the curve cases ................................ 208
  7.5.9 Some conjectures ........................................................................ 209
8 Symmetric quotient stacks and Heisenberg actions .................................... 213
  8.1 Introduction ...................................................................................... 213
    8.1.1 Generators of the Heisenberg algebra ....................................... 214
    8.1.2 Construction and results ......................................................... 215
    8.1.3 Conclusion .............................................................................. 216
    8.1.4 Organisation of the paper ...................................................... 216
  8.2 Proofs ............................................................................................... 217
    8.2.1 Proof of Lemma 8.1.2 ................................................................ 217
    8.2.2 Basic facts about equivariant functors ..................................... 218
    8.2.3 Combinatorial notations .......................................................... 219
    8.2.4 The functors \( P_3^{(n)} \) and \( Q_\alpha^{(n)} \) .................................. 219
    8.2.5 Proof of relation (8.6) ............................................................. 220
    8.2.6 Proof of relation (8.7) ............................................................. 220
  8.3 Further remarks ................................................................................. 222
    8.3.1 The Fock space ....................................................................... 222
    8.3.2 Left adjoints ........................................................................... 222
    8.3.3 Transposed generators ............................................................. 222
    8.3.4 Generalisations and variants .................................................... 223
    8.3.5 Open problems ....................................................................... 224
9 Remarks on the derived McKay correspondence for Hilbert schemes of
  points and tautological bundles .................................................................. 225
  9.1 Introduction ...................................................................................... 225
  9.2 Preliminaries ..................................................................................... 228
    9.2.1 General conventions .................................................................. 228
    9.2.2 Equivariant sheaves and derived categories ............................... 228
    9.2.3 Hilbert schemes of points and tautological sheaves ...................... 230
    9.2.4 Derived McKay correspondence .............................................. 230
    9.2.5 Combinatorial notations ........................................................... 231
    9.2.6 Scala’s theorem ....................................................................... 232
  9.3 Tautological bundles under the derived McKay correspondence ............ 233
    9.3.1 Various universal families and their geometry ............................ 233
    9.3.2 Tautological objects under the derived McKay correspondence ..... 234
  9.4 Extension groups .............................................................................. 237
  9.5 Tensor products and their Euler characteristic .................................... 240
9.5.1 Invariants of tensor products under the McKay correspondence . . . . . 240
9.5.2 Euler characteristic of tensor products of tautological bundles . . . . . 241
9.6 Further remarks . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 242
  9.6.1 Tautological bundles on Hilbert schemes of points on curves . . . . . . 243
  9.6.2 Wedge powers of tautological bundles of higher rank . . . . . . . . . . . 246
9.A Appendix: Computations with power series . . . . . . . . . . . . . . . . . . . . 246
Chapter 0

Introduction

Derived categories were first introduced as the adequate framework for derived functors in homological algebra. Over the last few decades the (bounded) derived category $D(X) := D^b(\text{Coh}(X))$ of coherent sheaves on a variety or, more generally, an orbifold $X$ was also recognised as an interesting geometric invariant which controls many other invariants such as cohomology and K-theory. One of the reasons for the recent interest in this category is a conjectured relationship to theoretical physics, more concretely to string theory, via the homological mirror symmetry; see [Kon95]. The derived category is also widely considered as the correct bridge between commutative and non-commutative algebraic geometry. From this point of view, the ‘spaces’ of non-commutative geometry are triangulated categories and the image of a variety $X$ in the non-commutative world is its derived category $D(X)$. The principal problems in the theory of derived categories may be summarised as follows.

**P1** Determine under which circumstances two smooth varieties or orbifolds $X$ and $X'$ have equivalent derived categories $D(X) \cong D(X')$ (*Fourier–Mukai partners*).

**P2** Describe the symmetries of the derived category (*groups of autoequivalences*).

**P3** Decompose the derived category into smaller pieces (*semi-orthogonal decompositions*).

**P4** Lift interesting structures of other invariants, such as cohomology and K-theory, to the level of the derived categories (*categorification*).

One should probably add the study of stability conditions, which has become very popular recently, as a fifth central problem. However, stability conditions only play a very minor role in this thesis. Hence, in this introduction, we will focus on the Problems **P1**, **P2**, **P3**, and **P4**.

Clearly, Problem **P4** is the most vague one but it often inspires progress towards the three more concrete goals. Very roughly, the state of the art concerning **P1**, **P2**, and **P3** for smooth projective varieties is as follows (we will come back to this in greater detail in the next section). For curves, everything is known. For surfaces, Problem **P1** is completely solved culminating in the work of Bridgeland and Maciocia [BM01] and, concerning **P2** and **P3**, a lot is known but there is still much work to do. In higher dimensions, complete results are only achieved for very special classes: For abelian varieties **P1**, **P2**, and **P3** are solved by work of Mukai [Muk81], Polishchuk [Pol96], and Orlov [Orl02] and, for varieties with ample or anti-ample canonical bundle, **P1** and **P2** are solved by Bondal and Orlov [BO01].
One main topic of this Habilitation thesis is the study of the derived category of Hilbert schemes of points on surfaces and generalised Kummer varieties. They are very interesting series of higher dimensional smooth varieties with various connections to combinatorics, representation theory, and theoretical physics; see [Nak99; Göt02] for an overview of some of these aspects. Their geometry is closely connected to that of the underlying surfaces. Hence, one can use the various results known for surfaces to study the derived categories of these higher dimensional varieties. The hope is that the results obtained for the derived categories of these special series allow to make reasonable conjectures, and ultimately prove them, concerning questions \( P_1, P_2, P_3, \) and \( P_4 \) in higher dimensions. In particular, Hilbert schemes of points on K3 surfaces and generalised Kummer varieties form a large part of the known examples of compact hyperkähler manifolds. Hence, one can view many results of this thesis as a contribution to the study of the derived category of hyperkähler manifolds.

The second, closely related, main theme of this thesis is the McKay correspondence. The derived McKay correspondence is a categorical version of the general philosophy, inspired from physics, that for a finite group \( G \) acting on a smooth variety \( M \), the geometry of the orbifold \( [M/G] \) and a crepant resolution \( Y \) should coincide in an appropriate sense.

In Section 0.1 of this introduction, we explain some of the background and make a serious attempt to give an overview of related work of other authors. In Section 0.2, we summarise the results of the author’s research papers which constitute this thesis.

Acknowledgements. The author thanks Ciaran Meachan, David Ploog, and Sönke Rollenske for useful comments on this introduction. Further acknowledgements are given at the beginning of the chapters, each of which consists of one research paper of the author.

0.1 Background, history, and state of the art

0.1.1 Derived functors and relative Duality

When studying a variety \( X \), one is often interested in vector bundles over this variety (the same holds for other geometric objects such as manifolds). The category \( \mathcal{V}B(X) \) of vector bundles, however, does not have very satisfactory properties. While morphisms of varieties induce pull-back functors on the level of vector bundles, they do not induce push-forwards. Furthermore, the category \( \mathcal{V}B(X) \) is not abelian, which means that kernels and cokernels do not exist in general. The solution of both problems is to pass to the bigger category \( \mathcal{C}oh(X) \) of coherent sheaves on \( X \). One can think of a coherent sheaf as a vector bundle where the fibre dimension is allowed to jump. The category \( \mathcal{C}oh(X) \) is abelian and, more or less, all geometric functors that one wants to define can be defined. In particular, for \( f: X \to Y \) a proper morphism, there is the push-forward \( f_*: \mathcal{C}oh(X) \to \mathcal{C}oh(Y) \). However, these functors do not preserve short exact sequences in general and this is the reason that one often has to consider derived functors.

Derived categories were introduced by Grothendieck and Verdier in the early 1960s as the adequate framework for these derived functors. The derived category of an abelian category \( \mathcal{A} \) is defined as \( \mathcal{D}(\mathcal{A}) := \mathcal{K}om(\mathcal{A})[\text{qis}^{-1}] \), the category of complexes localised by the class of quasi-isomorphisms, i.e., morphisms of complexes that induce isomorphisms on the level of the cohomology. This means that the objects of \( \mathcal{D}(\mathcal{A}) \) are the same as those of \( \mathcal{K}om(\mathcal{A}) \), namely complexes of objects of \( \mathcal{A} \). The morphisms in \( \mathcal{D}(\mathcal{A}) \) are the usual morphisms of complexes together with formal inverses of quasi-isomorphism. In particular, there is a functor \( \mathcal{K}om(\mathcal{A}) \to \mathcal{D}(\mathcal{A}) \) which turns every quasi-isomorphism into an honest isomorphism. The
original category $\mathcal{A}$ is embedded into the derived category by considering objects in $\mathcal{A}$ as complexes concentrated in degree zero. The derived category $D(\mathcal{A})$ is not abelian anymore but a triangulated category instead. This means that there is the autoequivalence $[1] : D(\mathcal{A}) \to D(\mathcal{A})$, given by the degree shift of complexes. Furthermore, there is a class of exact (or distinguished) triangles behaving similar to the class of short exact sequences in an abelian category. In particular, under the embedding $\mathcal{A} \to D(\mathcal{A})$, short exact sequences become distinguished triangles.

When we speak of the derived category of a variety $X$ (we always work over the field $\mathbb{C}$), we usually mean $D(X) := D^b(\text{Coh}(X)) \subset D(\text{Coh}(X))$, the full subcategory of complexes with bounded cohomology inside the derived category of coherent sheaves. Up to some details, all the geometric functors yield derived functors on the level of the derived categories. For example, for $f : X \to Y$ a proper morphism, there is the derived push-forward $Rf_* : D(X) \to D(Y)$ which is an exact functor of triangulated categories, which means that it commutes with the shift functor and preserves exact triangles. Furthermore, it recovers the higher direct image functors as, for every sheaf $F \in \text{Coh}(X) \subset D(X)$ and every $i \in \mathbb{Z}$, we have $H^i(Rf_*F) \cong R^if_*(F)$. However, the functor $Rf_*$ contains more information than all of the functors $R^if_*$ together. One very convenient consequence is that, given a second proper morphism $g : Y \to Z$, the information of the Leray spectral sequence

$$E_2^{p,q} = R^pg_*(R^if_*F) \quad \implies \quad F^{p+q} = R^{p+q}(g \circ f)_*F$$

can be summarised by the simple formula $Rg_* \circ Rf_* \cong R(g \circ f)_*$.

Note that, for a proper morphism $f : X \to Y$, the non-derived push-forward $f_* : \text{Coh}(X) \to \text{Coh}(Y)$ does not have a right adjoint in general (if it does, $f_*$ is already exact). A very deep result, known as Grothendieck duality or Grothendieck-Verdier duality, is that this changes if we pass to the derived functors. If $X$ and $Y$ are smooth (which is the case for most of the varieties that we will consider) the right adjoint is given by $f^! = Lf^*(\_ \otimes \omega_f)$ where the relative canonical bundle is $\omega_f = \omega_X \otimes f^*\omega_Y$. If we take $Y = \text{pt}$ to be the point, the derived push-forward can be identified with the sheaf cohomology functor $H^*(X, \_)$ and Grothendieck duality specialises to Serre duality.

The behaviour of smooth projective varieties regarding questions $\mathbf{P1, P2, P3}$ depends strongly on the positivity of the canonical bundle. The reason behind this is that Serre duality is an intrinsic feature of the derived category. Indeed, Serre duality can be reformulated by saying that the Serre functor $S_X = (\_ \otimes \omega_X[\dim X]) : D(X) \to D(X)$ fulfils the property

$$\text{Hom}_{D(X)}(E, F) \cong \text{Hom}(F, S_X(E))^\vee$$

for every $E, F \in D(X)$. Furthermore, the Serre functor is uniquely determined by this property due to the Yoneda Lemma. Hence, one can say that the derived category $D(X)$ sees the canonical bundle of $X$.

Problems $\mathbf{P1}$ and $\mathbf{P2}$ are completely solved for ample or antample canonical bundles but tend to become very complicated and deep for varieties with trivial (or close to trivial) canonical bundle. Problem $\mathbf{P3}$ has a simple answer for $\omega_X = \mathcal{O}_X$: there are no semi-orthogonal decompositions of $D(X)$ at all. On the other hand, there are usually plenty of interesting decompositions of the derived categories for Fano varieties and varieties of general type. We will say more about these, and other, phenomena in the following subsections.
0.1.2 Fourier–Mukai partners P1

Originally, the derived category was mainly seen as a tool which allows to deal with derived functors in a natural way and which is necessary for the formulation of relative Serre duality. Starting with the work of Mukai [Muk81] on derived duality for abelian varieties, the derived category began to be recognised as an interesting geometric invariant. Concretely, he proved that, for every abelian variety \( A \), there is an exact equivalence of categories \( D(A) \cong D(\hat{A}) \) where \( \hat{A} = \text{Pic}^0 A \) is the dual abelian variety. To appreciate this result, one should compare it to a theorem of Gabriel [Gab62] which says that the category of coherent sheaves determines a variety up to isomorphism:

\[
\text{Coh}(X) \cong \text{Coh}(Y) \iff X \cong Y.
\]

In contrast, note that \( A \not\cong \hat{A} \) in general. So it happens that the derived category identifies non-isomorphic varieties but, in all known cases, these varieties, which are then called Fourier–Mukai partners (sometimes FM partners for short), are still strongly related. In Mukai’s example, the Fourier–Mukai partners are moduli-spaces of line bundles on each other. Furthermore, the equivalence \( \Phi: D(A) \xrightarrow{\cong} D(\hat{A}) \) is realised by the universal family, the Poincaré line bundle \( P \in \text{Pic}(A \times \hat{A}) \). Namely, we have

\[
\Phi = p_\ast(q^\ast(\_ \otimes P))
\]

where \( p: A \times \hat{A} \to \hat{A} \) and \( q: A \times \hat{A} \to A \) are the projections. Note that, from now on, all functors are implicitly understood to mean their derived version. This means that \( p_\ast \) stands for the derived push-forward \( Rp_\ast: D(A \times \hat{A}) \to D(\hat{A}) \), \( q^\ast \) stands for \( Lq^\ast: D(A) \to D(A \times \hat{A}) \), and so on.

In some regard, the picture that Fourier–Mukai partners are moduli spaces of objects on each remains true in general due to the following theorem of Orlov.

**Theorem 0.1.1** ([Orl03]). Every equivalence \( \Phi: D(X) \xrightarrow{\cong} D(Y) \) between derived categories of smooth projective varieties is a Fourier–Mukai transform, which means that

\[
\Phi = \text{FM}_\mathcal{Q} := p_\ast(q^\ast(\_ \otimes \mathcal{Q})) \quad \text{for some } \mathcal{Q} \in D(X \times Y).
\]

Thus, we can regard \( Y \) as the parameter space of the fibres \( \mathcal{Q}_{|X \times \{y\}} \in D(X) \) which are objects over \( X \) and, conversely, \( X \) as the parameter space of the fibres \( \mathcal{Q}_{|(x) \times Y} \in D(Y) \). Furthermore, Orlov’s result guarantees that derived equivalences induce isomorphisms on the level of cohomology and K-theory. In other words, the derived category controls these other invariants.

**Fourier–Mukai partners with (anti-)ample canonical bundle**

In general, it is a hard problem to find the Fourier–Mukai partners of a given smooth projective variety \( X \). However, for extremal values of the positivity of the canonical bundles, the derived category still determines the variety.

**Theorem 0.1.2** ([BO01]). Let \( X, Y \) be smooth projective varieties and let \( \omega_X \) or \( \omega_X^{-1} \) be ample. Then:

\[
D(X) \cong D(Y) \iff X \cong Y.
\]
While the actual proof of that theorem is non-trivial, the reason behind its truth is easily summarised. As explained above, the derived category $D(X)$ sees the canonical bundle via its Serre functor. Furthermore, under the assumption that $\omega_X$ is ample, the variety $X$ is determined by the canonical ring $R(\omega_X) = \oplus_{k \geq 0} H^0(\omega_X^k)$ (and by its anti-canonical ring if $\omega_X^{-1}$ is ample).

**Derived equivalence of abelian varieties**

The second class of varieties where a complete description of Fourier–Mukai partners is known are abelian varieties, mainly due to work of Orlov [Orl02]. It turns out that, besides its dual $\hat{A}$, an abelian variety can have further Fourier–Mukai partners, all of which are abelian varieties too [HN11, Prop. 3.1], and Orlov gives a precise criterion for two abelian varieties to be derived equivalent.

**Derived equivalence in low dimension**

In the case of elliptic curves, Orlov’s criterion for derived equivalence of abelian varieties shows that there are no non-trivial FM partners (note that every elliptic curves is principally polarised which means that $E \cong \hat{E}$). Together with Theorem 0.1.2 this shows that, as mentioned above, there are no non-trivial FM partners for smooth projective curves at all.

For smooth projective surfaces there are examples of non-trivial FM partners but only for abelian surfaces, K3 surfaces and elliptic surfaces; see [BM01]. This confirms the picture that problem P1 is the most interesting for $\omega_X$ trivial (abelian and K3 surfaces) or fairly close to trivial (trivial along the fibres of an elliptic fibration).

In dimension 3, most of the literature concerning problem P1 concentrates on the case of Calabi–Yau varieties; see for example [Cal07], [BC09], [CT16], [HT16]. The main reason for the strong interest in this special case is Kontsevich’s homological mirror symmetry [Kon95]. Superstring theory only works well if space-time is 10-dimensional. Hence, many string-theorists assume that there is a 3-dimensional (6-dimensional over the real numbers) Calabi–Yau manifold hidden in every point of ordinary space-time. Drastically oversimplified, Kontsevich’s homological mirror symmetry conjecture says that the resulting physical theory only depends on the derived category of the Calabi–Yau 3-fold, not on the choice of the manifold itself.

**Derived equivalences of varieties with trivial canonical bundle**

The problem to determine all FM partners of a given smooth projective variety seems to be particularly rich for varieties with trivial canonical bundle. It follows easily from the fact that Serre duality is an intrinsic data of the derived category, that the class of varieties with trivial canonical bundle is **stable under derived equivalences**: If $D(X) \cong D(Y)$ holds and $\omega_X$ is trivial, so is $\omega_Y$.

The three basic subclasses of the class of varieties with trivial canonical bundle are abelian varieties, **strict Calabi–Yau varieties** (varieties with trivial canonical bundle and the property that $H^i(X, \mathcal{O}_X) = 0$ for $i \neq 0, \dim X$), and **hyperkähler** varieties, also known as **irreducible holomorphic symplectic varieties** (simply connected varieties with trivial canonical bundle such that $H^0(X, \Omega_X^2)$ is generated by a holomorphic symplectic form). Up to étale covers, every variety with trivial canonical bundle is a product of varieties from these three classes due to work of Bogomolov and Beauville [Bea83]. As shown by Huybrechts and Nieper-Wißkirchen
[HN11], the classes of abelian and hyperkähler varieties are stable under derived equivalences. The same is conjectured to hold for strict Calabi–Yau varieties. This is known to hold in dimension up to four; see [LP15], [Abu15].

0.1.3 Autoequivalences

Let $X$ be a smooth projective variety. We consider the group $\text{Aut}(D(X))$ of isomorphism classes of exact autoequivalences of the triangulated category $D(X)$. Concerning Problem P2, the determination of the group $\text{Aut}(D(X))$, there are only very few complete results. To the best of the authors knowledge, the only classes of smooth projective varieties where $\text{Aut}(D(X))$ is completely described are those with (anti-)ample canonical bundle [BO01], abelian varieties [Orl02], surfaces of general type whose canonical model has only $A_n$-singularities [IU05], toric surfaces [BP14], some elliptic surfaces [Ueh16], and K3 surfaces of Picard rank 1 [BB17a]. In some other cases, there are at least concrete conjectures predicting how $\text{Aut}(D(X))$ could look like, but in most cases the emphasis of the current research is on constructing new autoequivalences. The most successful construction method is in using twists along certain types of objects or functors as we will explain later.

Standard autoequivalences

There are always two types of autoequivalences which come from autoequivalences of the abelian category $\text{Coh}(X)$: Push-forwards (and pull-backs) along automorphisms of $X$ and tensor products $M_L := (\_ \otimes L$ by line bundles on $X$. In addition, there is always the shift autoequivalence $[1]$ along with its powers $[m]$ for $m \in \mathbb{Z}$. The subgroup of $\text{Aut}(D(X))$ spanned by these three types of autoequivalences is called the group of standard autoequivalences and denoted by

$$\text{Aut}_{\text{st}}(D(X)) := \langle [1], \{\varphi_\cdot \}_{\varphi \in \text{Aut} X}, \{M_L\}_{L \in \text{Pic} X} \rangle \cong \mathbb{Z} \times (\text{Aut} X \times \text{Pic} X).$$

**Theorem 0.1.3** ([BO01]). For $\omega_X$ ample or anti-ample, the standard autoequivalences are actually the only autoequivalences of the derived category:

$$\text{Aut}(D(X)) = \text{Aut}_{\text{st}}(D(X)).$$

In general, however, there are non-standard autoequivalences of the derived category. For example, if $A$ is a principally polarized abelian surface, i.e. $A \cong \tilde{A}$, the Fourier–Mukai transform along the Poincaré bundle $\text{FM}_PF \in \text{Aut}(D(A))$ sends skyscraper sheaves of points to line bundles and hence cannot be standard.

Spherical objects and twists

An important source of non-standard autoequivalences are twists along spherical objects as introduced by Seidel and Thomas [ST01]. For $E, F \in D(X)$, we consider the graded Hom-space

$$\text{Hom}^*(E, F) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}^i(E, F)[-i] \quad \text{where} \quad \text{Hom}^i(E, F) := \text{Hom}_{D(X)}(E, F[i]).$$

An object $E \in D(X)$ is called spherical if $E \otimes \omega_X \cong E$ and $\text{Hom}^*(E, F) \cong \mathbb{C}[0] \oplus \mathbb{C}[-\dim X]$. The latter graded vector space agrees with the cohomology $H^*(S^\dim X, \mathbb{C})$ of the sphere which
explains the notion. Using the triangulated structure of $D(X \times X)$, one can associate an autoequivalence $T_E \in \text{Aut}(D(X))$ to every spherical object, the spherical twist. Concretely, $T_E := \text{FM}_Q$ where $Q \in D(X \times X)$ is the unique object fitting into an exact triangle
\[ E \boxtimes E^\vee \xrightarrow{\cong} O_\Delta \rightarrow Q \rightarrow E \boxtimes E^\vee[1]. \]
The twist autoequivalence satisfies $T_E(E) = E[1 - \dim X]$ and acts as the identity on the right orthogonal complement
\[ E^\perp := \{ F \in D(X) \mid \text{Hom}^*(E,F) \neq 0 \}. \]
For $\dim X > 1$ it follows immediately that $T_E$ is non-standard (at least if $E^\perp$ is non-empty, which holds for all known examples but is an open question in general) since a standard autoequivalence could not shift two objects by different values.

The motivation for spherical objects and twists came from homological mirror symmetry. Namely, spherical twists are expected to correspond to Dehn twists around Lagrangian spheres on the symplectic side of mirror symmetry.

Let $X$ be a strict Calabi–Yau variety, which means that $\omega_X$ is trivial and $H^i(O_X) = 0$ for $i \neq 0, \dim X$. Then every line bundle on $X$, in particular $O_X$, is a spherical object. We have $T_{O_X} = \text{FM}_{I_\Delta[1-\dim X]}$. A second standard example is $O_C \in D(X)$ for $\mathbb{P}^1 \cong C \subset X$ a $(-2)$-curve on a surface.

### Autoequivalences on surfaces

For some, but not many, classes of surfaces, there is a complete description of the group of derived autoequivalences. The results of [IU05], [BP14], and [BB17a] can be summarised by saying that for surfaces of general type whose canonical model has only $A_n$-singularities, toric surfaces, and K3 surfaces of Picard rank 1, the group $\text{Aut}(D(X))$ is spanned by standard autoequivalences and twist along spherical objects. For the elliptic surfaces studied in [Ueh16], essential the only autoequivalence that has to be added is the FM transform along the relative Poincaré bundle of the fibration. In all the cases, there are often braid relations between the generators of $D(X)$. We will discuss the more general principle behind this in Subsection 0.1.7.

For arbitrary K3 surfaces $X$, there is at least a concrete conjecture of Bridgeland describing how the group $\text{Aut}(D(X))$ looks like. The exact formulation of Bridgeland’s conjecture involves the space of stability conditions which is beyond the scope of this introduction. However, the main concrete consequence of the conjecture for the group of autoequivalences is that $\text{Aut}(D(X))$ should again be generated by standard autoequivalences and twists along spherical objects.

### Generalisations of spherical objects

Spherical objects and twists have been generalised into various directions. Huybrechts and Thomas [HT06] introduced the notion of $\mathbb{P}$-objects, mainly in order to construct non-standard autoequivalences on hyperkähler manifolds. An object $E \in D(X)$ of the derived category of a smooth projective variety is a $\mathbb{P}^n$-object if $E \otimes \omega_X \cong E$ and there is an isomorphism of $\mathbb{C}$-algebras
\[ \text{Hom}^*(E,E) \cong H^*(\mathbb{P}^n,\mathbb{C}) \cong \mathbb{C}[t]/t^{n+1} \quad \text{with } \deg t = 2. \]
Again, there is an associated $\mathbb{P}$-twist $P_E \in \text{Aut}(D(X))$. It satisfies $P_E(E) \cong E[2n]$ and $P_{E|E^\perp} \cong \text{id}$. The standard examples of $\mathbb{P}$-objects are line bundles on compact hyperkähler manifolds
and structure sheaves of centres of Mukai flops, i.e. \( \mathcal{O}_Z \in \mathcal{D}(X) \) where \( \dim X = 2n \) and \( \mathbb{P}^n \cong Z \subset X \) with \( N_{Z/X} \cong \Omega_{\mathbb{P}^n} \).

Another very important generalisation of spherical objects is the notion of spherical functors; see [Ron06], [AL13]. It was inspired by and subsumes the EZ spherical objects of Horja [Hor05] and the fat spherical objects of Toda [Tod07]. Let \( F = \mathcal{F}M_Q : \mathcal{D}(Z) \to \mathcal{D}(X) \) be a Fourier–Mukai transform between the derived categories of smooth projective varieties. It follows by Grothendieck duality that \( F \) has left and right adjoints \( F^L, F^R : \mathcal{D}(X) \to \mathcal{D}(Z) \). Let \( \eta : \text{id} \to F^RF \) be the unit of adjunction. The first condition on \( F \) to be a spherical functor is that the endofunctor \( C : \mathcal{D}(Z) \to \mathcal{D}(Z) \), called the cotwist of \( F \), that fits into the exact triangle of functors

\[
\text{id} \xrightarrow{\eta} F^RF \to C \to [1]
\]

is an autoequivalence. Note that, in general, exact triangles are not functorial. However, in our case all the functors involved are Fourier–Mukai transforms. Hence it makes sense to talk about exact triangles of functors using the triangulated structure of \( \mathcal{D}(Z \times X) \), the category where the corresponding FM kernels live. One can still speak about spherical functors in the larger context of dg-enhanced triangulated categories, but this is not necessary for the present thesis. The second condition for \( F \) to be a spherical functor is that there is an equivalence \( F^R \cong CF^L \) relating the right and left adjoints, which can be reformulated in terms of Serre functors as the condition \( S_X FC \cong FS_Z \). For \( Z = \text{pt} \) a point, the Fourier–Mukai kernel \( Q \in \mathcal{D}(\text{pt} \times X) \cong \mathcal{D}(X) \) is a spherical object if and only if \( F = \mathcal{F}M_Q : \mathcal{D}(\text{pt}) \to \mathcal{D}(X) \) is a spherical functor. One can think of spherical functors as relative spherical objects which is nicely illustrated by one of the main examples: the pull back along a Calabi–Yau fibration. Analogously to the case of spherical objects, there is a spherical twist \( T_F \in \text{Aut}(\mathcal{D}(X)) \) defined by the exact triangle

\[
FF^R \xrightarrow{\varepsilon} \text{id} \to T_F \to FF^R[1]
\]

where \( \varepsilon \) is the counit of adjunction. In many examples of spherical functors, the triangle (1) splits so that we have \( F^RF \cong \text{id} \oplus C \). In this case we call \( F \) split spherical.

Segal [Seg16] proved that every autoequivalence \( \Phi \in \text{Aut}(\mathcal{D}(X)) \) can be realised as a spherical twists \( \Phi = T_F \) along a spherical functor \( F : \mathcal{C} \to \mathcal{D}(X) \) where \( \mathcal{C} \) is a suitable dg-enhanced triangulated category. However, the category \( \mathcal{C} \) is usually more complicated than \( \mathcal{D}(X) \) itself. Accordingly, in practice, one cannot hope to find the spherical functor \( F : \mathcal{C} \to \mathcal{D}(X) \) before knowing the autoequivalence \( \Phi \). Hence, it is good to have other construction methods of autoequivalences, besides spherical functors, available.

One of the most recent developments in the construction of non-standard autoequivalences is the introduction of \( \mathbb{P} \)-functors which is a simultaneous generalisation of \( \mathbb{P} \)-objects and of split spherical functors due to Addington [Add16] (there is also a similar definition by Cautis [Cau12]). A Fourier–Mukai transform \( F : \mathcal{D}(Z) \to \mathcal{D}(X) \) is called a \( \mathbb{P}^n \)-functor if there exists an autoequivalence \( D \in \text{Aut}(\mathcal{D}(Z)) \), called the \( \mathbb{P} \)-cotwist of \( F \), such that

\[
F^RF \cong \text{id} \oplus D \oplus D^2 \oplus \cdots \oplus D^n,
\]

we have \( F^R \cong D^nF^L \), or equivalently \( S_X F^R D^n \cong FS_Z \), and the monad structure of \( F^RF \) fulfils a certain condition whose technical formulation we omit. For \( Z = \text{pt} \) and \( D = [-2] \), the definition of a \( \mathbb{P}^n \)-functor specialises to that of a \( \mathbb{P}^n \)-object, and the condition on the monad structure becomes the condition that \( \text{Hom}^*(Q, Q) \cong H^*(\mathbb{P}^n, \mathcal{C}) \) not only as graded vector
spaces but as algebras. Yet again, there is an associated $\mathbb{P}$-twist $P_F \in \text{Aut}(D(X))$ satisfying

$$P_F F \cong FH^{n+1}[2], \quad P_{F[\text{im} F]^{\perp}} \cong \text{id} \quad \text{where} \quad (\text{im} F)^{\perp} = \ker(F^R) = \{ A \in D(X) \mid F^R(A) = 0 \}.$$  

(2)

Note that a $\mathbb{P}^1$-functor is the same as a split spherical functor. In this case, the $\mathbb{P}$-twist is the square of the spherical twist: $P_F \cong T_F^2$.

**Derived categories of hyperkähler manifolds**

Many of the applications of $\mathbb{P}$-functors involve hyperkähler manifolds. The main example of a $\mathbb{P}^{n-1}$-functor in [Add16] is the Fourier–Mukai transform $F = FM_{\Xi}: D(X) \to D(X[n])$ along the ideal sheaf of the universal family $\Xi \subset X \times X[n]$ of length $n$ subschemes, for $X$ a K3 surface and $n \geq 2$. Similarly, for $A$ an abelian surface, the FM transform along the universal ideal sheaf of the generalised Kummer variety gives a $\mathbb{P}^{n-1}$-functor $D(A) \to D(K_nA)$ for $n \geq 2$; see [Mea15].

This fits into the following more general picture: All known examples of compact hyperkähler manifolds are deformation equivalent to (resolutions of) moduli spaces of sheaves on K3 or abelian surfaces with the Hilbert schemes of points and the generalised Kummer varieties being the most prominent representatives of the deformation equivalence classes. It is conjectured (or, at least, considered as an interesting working hypothesis) that every compact hyperkähler manifold can be realised as a moduli space of objects in a 2-dimensional Calabi–Yau category (CY2 category), i.e. a triangulated category with the same formal properties as a K3 or abelian surfaces (technically, this means that shift by 2 is the Serre functor of the category). This category can then be viewed as a non-commutative K3 or abelian surfaces. Furthermore, it is conjectured that the CY2 category and the associated $2n$-dimensional hyperkähler are connected by a $\mathbb{P}^{n-1}$-functor with $\mathbb{P}$-cotwist $D = [-2]$.

A case where this works beautifully is that of the variety $F(Y)$ of lines on a smooth cubic fourfold $Y \subset \mathbb{P}^5$. The variety $F(Y)$ is a hyperkähler fourfold deformation equivalent to the Hilbert scheme of points on a K3 surface but, in general, not isomorphic to a Hilbert scheme; see [BD85]. However, due to Kuznetsov [Kuz10], there is always a K3 category $\mathcal{A}_Y$ (a CY2 category with the same Hochschild homology as a K3 surface) contained in $D(Y)$ which is conjectured to be strongly related to the question whether or not $Y$ is rational. Addington [Add16, Sect. 5] proved that the Fourier–Mukai transform $D(Y) \to D(F(Y))$ along the universal family of lines restricts to a split spherical functor $\mathcal{A}_Y \to D(F(Y))$ with cotwist $[-2]$. Further partial results in the direction of the above conjecture are given by Markman and Mehrotra [MM15], Kuznetsov [Kuz15], and Addington, Donovan, and Meachan [ADM16].

**Equivalences induce autoequivalences**

Another large source of autoequivalences of the derived categories of smooth projective varieties are equivalences $\Phi : D(X) \xrightarrow{\cong} D(Y)$ between Fourier–Mukai partners. Indeed, such an equivalence is of the form $\Phi = FM_Q$ for some $Q \in D(X \times Y)$. Interpreting the Fourier–Mukai kernel as an object of $D(Y \times X)$ instead, we get the associated Fourier–Mukai transformation $\Psi = FM_{Q^R}: D(Y) \to D(X)$ which is again an equivalence but, usually, not the inverse of $\Phi$. Hence, we get an autoequivalence $\Psi \Phi \in \text{Aut}(D(X))$ which is non-trivial, and even non-standard, in general. Hence, given a derived equivalence $\Phi$, one can ask how to describe the induced autoequivalence $\Psi \Phi$ intrinsically in terms of $D(X)$, for example as one of the
twists introduced above. This question is particularly interesting in the context of birational geometry; see Subsection 0.1.5 below.

0.1.4 Semi-orthogonal decompositions and exceptional sequences P3

It is a very common approach throughout various mathematical theories to study the objects of interest by decomposing them until one reaches irreducible pieces. Our objects of interest are the triangulated categories $D(X)$, usually for $X$ a smooth projective variety. There is indeed a well-defined notion of a direct sum of triangulated categories. However, this only reflects the decomposition of $X$ into connected components. More precisely, if $D(X) = T_1 \oplus T_2$ for triangulated categories $T_1$, $T_2$, we already have $T_i = D(X_i)$ with $X = X_1 \coprod X_2$.

It turns out that a much more interesting notion is that of a semi-orthogonal decomposition $D(X) = \langle T_1, T_2 \rangle$. We omit the technical details of the definition, but the main difference is that we only require $\text{Hom}(T_2, T_1)$ to vanish while, in contrast, for a direct sum decomposition $D(X) = T_1 \oplus T_2$ we also have $\text{Hom}(T_1, T_2) = 0$. However, the notion of a semi-orthogonal decomposition still guarantees that for every object $A \in D(X)$ there is a unique triangle

$$A_2 \to A \to A_1 \to A_2[1]$$

with $A_i \in T_i$, which makes this into a reasonable notion of a decomposition. In particular, the two factors of the semi-orthogonal decomposition $D(X) = \langle T_1, T_2 \rangle$ determine each other as $T_1 = T_2^\perp$ and $T_2 = -T_1$. Of course, there are possibly further semi-orthogonal decompositions of the triangulated subcategories $T_i$ giving rise to semi-orthogonal decompositions of $D(X)$ of higher length.

Every fully faithful embedding $D(Z) \hookrightarrow D(X)$ of derived categories of smooth projective varieties induces a semi-orthogonal decomposition $D(X) = \langle D(Z), T_2 \rangle$ due to a generalised version of Theorem 0.1.1. For $Z = \text{pt}$ a point such an embedding corresponds to an exceptional object $E \in D(X)$ which means that $\text{Hom}^*(E, E) = \mathbb{C}[0]$. Many fully faithful embeddings $D(Z) \hookrightarrow D(X)$ are constructed by Kuznetsov’s homological projective duality [Kuz07].

Irreducible derived categories and the canonical bundle

An example of a triangulated category which is irreducible with regard to the notion of semi-orthogonal decomposition is $D(X)$ for $X$ a smooth projective variety with trivial canonical bundle. Indeed, Serre duality implies that $\text{Hom}(T_1, T_2) = 0$ if and only if $\text{Hom}(T_2, T_1) = 0$ for triangulated subcategories $T_i \subset D(X)$. Hence, in this case a semi-orthogonal decomposition would already give a direct sum decomposition which cannot happen for $X$ connected. A more general criterion for irreducibility of $D(X)$ in terms of $\omega_X$ is given by Kawatani and Okawa [KO15]. In particular, there are no semi-orthogonal decompositions of $D(X)$ if the base-point locus of $\omega_X$ is zero-dimensional. It follows that the only smooth projective curve with a non-trivial semi-orthogonal decomposition is $\mathbb{P}^1$ (which was already proven earlier by Okawa [Oka11]). Furthermore, it rules out semi-orthogonal decompositions on many types of minimal surfaces, such as bielliptic surfaces.

Derived categories of Fano varieties

On the other hand, Fano varieties and varieties of general type tend to have a lot of semi-orthogonal decompositions. For a Fano variety $X$ (i.e. a smooth projective variety with $\omega_X^{-1}$
ample), every line bundle is an exceptional object by Kodaira’s vanishing theorem, realising \( D(\text{pt}) \) as a semi-orthogonal factor of \( D(X) \). The abundance of semi-orthogonal decompositions on Fano varieties lead to the question (originally posed by Bondal, see also [BBF16]) whether every smooth projective variety is a Fano visitor: Given a smooth projective variety \( Y \) is there a Fano variety (or at least a Fano orbifold) \( X \) such that there exists a fully faithful embedding \( D(Y) \hookrightarrow D(X) \)? An affirmative answer would be a very interesting theoretical result. Probably exaggerating its impact, one could say that this would reduce non-commutative geometry to the study of Fano varieties. More concretely, realising a variety as a Fano visitor can be very helpful for understanding the geometry of \( Y \) since one can view it as a moduli space of objects on its Fano host \( X \). Varieties which are known to be Fano visitors include complete intersections in projective space [Kie+15] and curves [KL15].

**Phantom categories**

Another very interesting phenomenon, occurring primarily on varieties of general type, is that of a phantom category; see [BBS13], [Böh+15], [GS13], [Gal+15], [AO13], [Cot15], [Lee15]. They are admissible subcategories (which means factors of a semi-orthogonal decomposition) of \( D(X) \) which are non-zero but their two typical categorical invariants, Hochschild homology and Grothendieck group, vanish. Phantom categories are often related to the deformation theory of surfaces of general type. Furthermore, they seem to play an interesting role in homological mirror symmetry; see [FHK14].

**Exceptional sequences**

Not surprisingly, a case of particular interest of a semi-orthogonal decomposition \( D(X) = \langle T_1, \ldots, T_n \rangle \) is when all the factors of the decomposition are given by the derived category of a point (which is equivalent to the derived category of \( \mathbb{C} \)-vector spaces). This case corresponds to a full exceptional sequence of \( D(X) \). An exceptional sequence consists of exceptional objects \( E_1, \ldots, E_n \in D(X) \) satisfying the semi-orthogonality assumption \( \text{Hom}^*(E_i, E_j) = 0 \) for \( i > j \). The sequence is called full if it generates \( D(X) \) in the sense that the only full triangulated subcategory of \( D(X) \) containing all the \( E_i \) is \( D(X) \) itself. We call the sequence \( E_1, \ldots, E_n \) strong if it satisfies that \( \text{Hom}^*(E_i, E_j) \) is concentrated in degree zero (also for \( i < j \)). If \( E_1, \ldots, E_n \in D(X) \) is a full strong exceptional sequence, one obtains an exact equivalence \( D(X) \cong D(\text{Mod}(A)) \) to the bounded derived category of finitely generated modules over the ring \( A := \text{End}(E) \) where \( E := \oplus_{i=1}^n E_i \in D(X) \); see [Bon89]. The ring \( A \) is a path algebra of a quiver with relations whose vertices correspond to the members of the exceptional sequence. This phenomenon gives a very interesting connection between algebraic geometry and representation theory. It is a far reaching generalisation of Beilinson’s description [Bel78] of the derived category of the projective space \( \mathbb{P}^n \). Later, Kapranov [Kap88] proved that Grassmannians, flag manifolds, quadrics, and some other rational homogeneous spaces have full strong exceptional collections. This led to the conjecture that the same holds for every rational homogeneous space; see [Böh06] for some more recent work on this conjecture.

The general belief is that varieties whose derived categories have full exceptional sequences are ‘strongly rational’ in some sense. There are concrete conjectural criteria for \( D(X) \) to have a full exceptional collection in terms of the image of \( X \) in the Grothendieck ring of varieties and its motive. Further examples of rational varieties known to have full exceptional sequences involve rational surfaces [HP11] and toric varieties [Kaw06], [Kaw13].
0.1.5 Birational geometry and derived categories

Another intriguing aspect of derived categories are connections, some of them conjectured, some of them proven, to birational geometry. Many of them can be summarised under the homological minimal model program. The basic idea is that running the minimal model program for a variety $X$ should correspond to minimising the derived category $D(X)$. In particular, the derived category $D(X')$ of a minimal model of $X$ should embed into $D(X)$. The first basic result in this direction is the description of the derived category of a blow-up $\pi: \tilde{X} \to X$ in a smooth centre due to Orlov [Orl92]. In particular, the pull-back $\pi^*: D(X) \to D(\tilde{X})$ is fully faithful.

One concrete expected phenomenon is the $DK$ Hypothesis of Bondal, Orlov, and Kawamata; see, for example [Kaw16, Sect. 2]. It concerns a birational correspondence

$$
\begin{array}{ccc}
Z & \xrightarrow{q} & X \\
\downarrow & & \downarrow \pi \\
X' & \xleftarrow{p} & X
\end{array}
$$

of smooth varieties. Then we should have:

- A fully faithful embedding $D^b(X) \hookrightarrow D^b(X')$, if $q^*K_X \leq p^*K_{X'}$.
- A fully faithful embedding $D^b(X') \hookrightarrow D^b(X)$, if $q^*K_X \geq p^*K_{X'}$.
- An equivalence $D^b(X') \cong D^b(X)$, in the flop case $q^*K_X = p^*K_{X'}$.

This is proven in many instances; see [BO95], [Bri02], [Kaw02], [Nam03].

Given an equivalence $\Phi: D(X) \cong D(X')$ between varieties related by a flop one can consider the flop-flop autoequivalence $\Psi\Phi \in \text{Aut}(D(X))$ where $\Psi: D(X') \cong D(X)$ is the autoequivalence with the same FM kernel as $\Phi$ but in the reverse direction; compare Section 0.1.3. Often, the centre of the birational morphism $q: Z \to X$ gives rise to spherical or $\mathbb{F}$-functors and the autoequivalence $\Psi\Phi$ can be described in terms of the associated twists. This is known as the flop-flop-twist principle and proven in many instances; see [Tod07], [BB15], [DW16], [DW15], [ADM15].

0.1.6 McKay correspondence

Canonical surface singularities are exactly the quotient singularities of the form $\mathbb{A}^2/G$ for a finite subgroup $G \subset \text{SL}(2, \mathbb{C})$. These kind of singularities, also known as Kleinian singularities, are in one-to-one correspondence with the ADE Dynkin diagrams. This correspondence is realised by considering the intersection graph of the exceptional ($-2$)-curves of the minimal resolution $Y \to \mathbb{A}^2/G$.

McKay [McK80] observed that there is a bijection between the non-trivial irreducible representations of the group $G$ and the irreducible components of the exceptional divisor of the resolution $Y$. This bijection relates the decompositions of the tensor products of the representations into irreducibles to the intersection graph of the exceptional divisor.

Gonzales-Sprinberg and Verdier [GV83] gave a conceptual explanation of this correspondence by constructing an isomorphism $K(Y) \cong K_G(\mathbb{A}^2)$ between the Grothendieck group of the minimal resolution $Y$ and the Grothendieck group of $G$-equivariant vector bundles on $\mathbb{A}^2$.

One way to ask for a generalisation of this picture is: Given a finite group $G$ acting on a
smooth variety $M$, is there a resolution $\mu: Y \to M/G$ of singularities which is minimal in an appropriate sense and such that the geometry of $Y$ reflects the representation theory of $G$ and the way the group acts on $M$?

It turned out that a very promising candidate for such a resolution is the $G$-Hilbert scheme $\text{Hilb}^G(M)$ which is (an irreducible component of) the moduli space of $G$-clusters on $M$; see [IN96], [IN00]. A $G$-cluster is a zero-dimensional $G$-invariant subscheme $Z \subset M$ such that the space of functions $O(Z)$, as a $G$-representation, equals the regular $G$-representation $\mathbb{C}[G]$. Every free $G$-orbit in $M$ together with its reduced subscheme structure is a $G$-orbit but there are also non-reduced $G$-clusters supported on non-free $G$-orbits. There is indeed a birational morphism $\tau: \text{Hilb}^G(M) \to M/G$, the $G$-Hilbert-Chow morphism. Let $[Z] \in \text{Hilb}^G(M)$ be the point representing a $G$-cluster $Z \subset M$. Then $\tau([Z]) \in M/G$ is the point of the quotient lying under the $G$-orbit on which $Z$ is supported. It follows easily from an inspection of Gonzales-Sprinberg’s and Verdier’s proof that the minimal resolution $Y \to \mathbb{A}^2/G$ for $G \subset \text{SL}(2, \mathbb{C})$ can be identified with $\text{Hilb}^G(\mathbb{A}^2)$.

**Theorem 0.1.4 ([BKR01]).** Let $G$ be a finite group acting on a smooth quasi-projective variety $M$ and let $\tau: \text{Hilb}^G(M) \to M/G$ be the $G$-Hilbert-Chow morphism. Assume that:

(i) The canonical bundle $\omega_M$ is locally $G$-trivial: For every $x \in M$, the isotropy subgroup $G_x \subset G$ acts trivially on the fibre $\omega_M(x)$.

(ii) The fibres of $\tau$ are not too big: $\dim(\text{Hilb}^G(M) \times_{M/G} \text{Hilb}^G(M)) \leq \dim(M) + 1$.

Then $\tau: \text{Hilb}^G(M) \to M/G$ is a crepant resolution, which means that $\text{Hilb}^G(M)$ is smooth and $\tau^*\omega_{M/G} \cong \omega_{\text{Hilb}^G(M)}$, and

$$\Phi := p_\ast q^\ast: D(\text{Hilb}^G(M)) \to D_G(M)$$

is an equivalence where

$$\text{Hilb}^G(M) \xleftarrow{q} Z \xrightarrow{p} M$$

are the projections from the universal family of $G$-clusters $Z \subset \text{Hilb}^G(M) \times M$.

The equivariant derived category $D_G(M)$ on the right-hand side of the derived McKay equivalence $\Phi: D(\text{Hilb}^G(M)) \cong D_G(M)$ is defined as $D_G(M) := D^b(\text{Coh}_G(M))$ where $\text{Coh}_G(M)$ is the abelian category of $G$-equivariant coherent sheaves. This category can be canonically identified with the category of coherent sheaves on the quotient stack (or, depending on the preferred notion, quotient orbifold):

$$\text{Coh}_G(M) \cong \text{Coh}([M/G]) \quad , \quad D_G(M) \cong D([X/G]).$$

This shows that we can interpret the derived McKay equivalence as an instance of a stacky/orbifold version of the DK hypothesis by considering the following flop diagram of orbifolds

$$
\begin{array}{c}
\text{[Z/G]} \\
\downarrow q \\
\bar{Y} \overset{\sim}{\longrightarrow} [X/G] \overset{\pi}{\longrightarrow} X/G.
\end{array}
$$
0.1.7 **Categorification P4**

The derived category $D(X)$ controls other invariants of the smooth projective variety $X$ such as the Grothendieck group $K(X)$ or the cohomology $H^*(X, \mathbb{C})$. The reason is that, by Theorem 0.1.1, every autoequivalence is a Fourier–Mukai transform. Hence, one can use its FM kernel in order to induce correspondences on the level of the Grothendieck groups and cohomology which are isomorphisms. There are also very interesting, though more subtle, connections of $D(X)$ to the image of $X$ in the Grothendieck ring of varieties and its motive; see for example [Tab15], [KS16]. In particular, the motives with rational coefficients of Fourier–Mukai partners only differ by Tate twists; see [Orl05].

Hence, whenever other invariants such as the Grothendieck group or the cohomology of a variety carry an interesting additional structure, it is a very natural problem to lift this structure in an appropriate way to the level of the derived category. This problem is known as *categorification*.

One principal idea is that, whenever you see a canonical direct sum decomposition of the cohomology or K-theory, you look for a semi-orthogonal decomposition on the derived category that induces the decomposition. A basic example is the semi-orthogonal decomposition

$$D(\tilde{X}) = \langle D(X), D(Z), \ldots, D(Z) \rangle_{(c-1) \text{ times}}$$

where $\tilde{X}$ is the blow-up of $X$ in the smooth centre $Z \subset X$ of codimension $c$; see [Orl92].

The second principal idea is that, whenever you see the action of a group or an algebra on the cohomology or K-theory, you look for a collection of Fourier–Mukai transforms which descends to the action of a set of generators of the group or algebra (naive categorification). It would be nice if the relations between the generators can already be seen as isomorphisms between compositions of the FM-transforms (weak categorification) and even nicer if these isomorphisms themselves can be organised in an interesting way (strong categorification).

An example is given by an $A_n$-chain of $(-2)$-curves on a surface $X$. Then reflection on the hyperplanes orthogonal to the classes of these curves define an action of $S_n$ on the cohomology $H^*(X, \mathbb{C})$. This lifts to a categorical action of the braid group $B_{n+1}$ on $D(X)$ given by the spherical twists along the structure sheaves of the curves; see [ST01].

0.1.8 **Derived categories of Hilbert schemes of points on surfaces**

One might have noticed that a large part of the results on derived categories summarised above concern varieties of low dimension. In higher dimensions not much is known besides for abelian varieties and varieties with (anti-)ample canonical bundle. One approach, followed by various authors including myself, is to study Hilbert schemes of points on surfaces, where one can use the broad knowledge concerning derived categories on surfaces. The hope is that, with results for this class of examples as an inspiration, one might be able to make more general statements regarding derived categories of higher-dimensional varieties.

**Definition and basic properties**

Given a smooth quasi-projective variety $X$ and a number $n \in \mathbb{N}$, the Hilbert scheme (also known as *Douady space*) of $n$ points on $X$ is defined as the fine moduli space of zero-dimensional length $n$ subschemes of $X$. Sending a length $n$ subscheme to its weighted support
defines the *Hilbert-Chow morphism* \( \mu: X^{[n]} \to X^{(n)} := X^n/\Sigma_n \) to the symmetric product of the variety. This makes \( X^{[n]} \) a resolution of the quotient singularities of \( X^{(n)} \). In particular, the Hilbert schemes of points on surfaces are always smooth of dimension \( \dim X^{[n]} = 2n \); see [Fog68]. For \( X \) a smooth variety of higher dimension, \( X^{[n]} \) is smooth only for \( n \leq 3 \).

As already mentioned in Section 0.1.3, if \( X \) is a K3 surface, the associated Hilbert scheme is one of the rare series of examples of a compact hyperkähler variety (also known as an irreducible holomorphic symplectic variety). A second series of examples are the generalised Kummer varieties \( K_nA \) associated to an abelian variety \( A \). They are the codimension 2 subvarieties of \( A^{[n+1]} \) whose points correspond to length \( n + 1 \) subschemes of \( A \) whose weighted support adds up to zero.

**The cohomology of \( X^{[n]} \): Nakajima operators and the Heisenberg action**

Götsche [Göt90] gave a formula for the Betti numbers of the Hilbert schemes, i.e. the dimensions of the graded pieces of \( H^*(X^{[n]}, \mathbb{Q}) \), in terms of the Betti numbers of the surface \( X \). It turned out that they coincide with the dimensions of the graded pieces of a certain irreducible representation, called the *Fock space*, of the Heisenberg Lie algebra \( \mathfrak{h}_V \) associated to the cohomology \( V := H^*(X, \mathbb{Q}) \) of the surface \( X \).

The Lie algebra \( \mathfrak{h}_V \) associated to a (graded) vector space \( V \) together with a (graded) symmetric bilinear form \( \langle \_, \_ \rangle \) is, as a vector space, given by a copy of \( V \) for every non-zero integer together with one extra base vector \( c \):

\[
\mathfrak{h}_V := \mathbb{Q} \cdot c \oplus \bigoplus_{n \in \mathbb{Z}\setminus\{0\}} V.
\]

For \( \beta \in V \), we denote its copy in the \( n \)-th direct summand \( V \) of \( \mathfrak{h}_V \) by \( a_\beta(n) \). The Lie bracket of \( \mathfrak{h}_V \) is defined by setting \( c \) to be central, which means \( [c, h] = 0 \) for every \( h \in \mathfrak{h}_V \), and

\[
[a_\alpha(n), a_\beta(m)] := \delta_{m,-n} n(\alpha, \beta) \cdot c = \begin{cases} 0 & \text{if } m \neq -n \\ n(\alpha, \beta) \cdot c & \text{if } m = -n. \end{cases}
\]

In the case of the cohomology \( V = H^*(X, \mathbb{Q}) \), the bilinear form one considers is the intersection pairing \( \langle \alpha, \beta \rangle = \int_X \alpha \cap \beta \).

Inspired by Götsche’s formula, Nakajima [Nak97] and Grojnowski [Gro96] gave geometric constructions of an action of \( \mathfrak{h}_V \) on the cohomology space \( \mathbb{H}_X := \bigoplus_{\ell \geq 0} H^\ell(X^{[\ell]}, \mathbb{Q}) \). For Nakajima’s construction, the incidence schemes

\[
Z^{\ell,n} = \{ (x, \xi, \xi') \mid \xi \subseteq \xi', \xi \text{ and } \xi' \text{ only differ in } x \} \subset X \times X^{[\ell]} \times X^{[n+\ell]}
\]

play a crucial role.

**The Hilbert scheme of points as \( G \)-Hilbert schemes**

Given a smooth quasi-projective surface \( X \), there is a birational morphism \( X^{[n]} \to \text{Hilb}^{\Sigma_n}(X^n) \) where \( \Sigma_n \) acts by permutation of the factors on \( X^n \). It identifies the reduced length \( n \) subschemes of \( X \) with the free \( \Sigma_n \)-orbits on \( X^n \) by mapping \( \{x_1, \ldots, x_n\} \mapsto \Sigma_n \cdot (x_1, \ldots, x_n) \). In his seminal work [Hai01], Haiman extended this to an isomorphism \( X^{[n]} \cong \text{Hilb}^{\Sigma_n}(X^n) \). His main application was a proof of two important conjectures in algebraic combinatorics: the MacDonald Positivity Conjecture and the \( n! \) Conjecture. For us, however, a very interesting
consequence is that this makes the Hilbert scheme of points accessible to the derived McKay correspondence. Indeed, one can check that the assumptions of Theorem 0.1.4 are satisfied. Hence, we get an equivalence
\[
\Phi: D(X[n]) \xrightarrow{\cong} D_{\mathcal{E}_n}(X^n).
\]
This equivalence allows to make various interesting constructions for the equivariant derived category \(D_{\mathcal{E}_n}(X^n)\) and translate them into results concerning the derived category \(D(X[n])\).

**Results on** \(D(X[n])\)

The first basic step into the direction of the study of problems \(P1\) and \(P2\) for the derived category of Hilbert schemes of points is the following theorem of Ploog.

**Theorem 0.1.5 ([Plo07]).** Let \(X, Y\) be smooth projective surfaces. Then
\[
D(X) \cong D(Y) \implies D(X[n]) \cong D(Y[n]).
\]

Furthermore, for every smooth projective surface \(X\), there is an injective group homomorphism
\[
\text{Aut}(D(X)) \hookrightarrow \text{Aut}(D(X[n])).
\]

The proof crucially uses the derived McKay correspondence \(D(X[n]) \cong D_{\mathcal{E}_n}(X^n)\): By Orlov’s Theorem 0.1.1 every autoequivalence \(D(X) \xrightarrow{\cong} D(Y)\) is a Fourier–Mukai transform \(FM_P\) for some object \(P \in D(X \times Y)\). One can show that its box power \(P^{\otimes n} \in D_{\mathcal{E}_n}(X^n \times Y^n)\) induces an equivalence \(D_{\mathcal{E}_n}(X^n) \xrightarrow{\cong} D_{\mathcal{E}_n}(Y^n)\).

Similarly using the box product, one can produce \(\mathbb{P}^n\)-objects on \(D(X[n])\) out of spherical objects on the surface \(X\); see [PS14].

Note that there is no general construction analogous to Theorem 0.1.5 known for generalised Kummer varieties. In particular, for \(A\) and \(B\) abelian surfaces and \(n \geq 2\) it is an open problem whether or not
\[
D(A) \cong D(B) \implies D(K_nA) \cong D(K_nB).
\]

However, there is an affirmative answer for \(n = 1\); see [Hos+03], [Ste07].

In regard of the categorification problem \(P4\) and the results on the cohomology of the Hilbert schemes described in Section 0.1.8, it is desirable to construct a categorical action of the Heisenberg algebra on the derived categories of Hilbert schemes of points on surfaces. This was achieved by Cautis and Licata [CL12] in the case that the surface \(X\) is a minimal resolution of a Kleinian singularity \(\mathbb{A}^2/G\). This Heisenberg action also induces further non-standard autoequivalences of \(D(X^[n])\); see [CLS14]. The construction of [CLS14] uses the fact that the structure sheaves of the exceptional curves of the resolution \(X \rightarrow \mathbb{A}^2/G\) are spherical objects and can be seen as a generalisation of the construction of [PS14].

The derived categories of the Hilbert scheme \(X^[n]\) and of its underlying surface \(X\) are always connected by two canonical functors \(F, F'': D(X) \to D(X^[n])\), namely the FM transforms along the ideal sheaf \(F = FM_{\mathbb{P}^2}\) and along the structure sheaf \(F'' = FM_{\mathbb{O}_{\mathbb{P}^2}}\) of the universal family of length \(n\) subschemes \(\Xi \subset X \times X^[n]\).

In Section 0.1.3, we already discussed the fact that, for \(X\) a K3 surface, \(F\) is a \(\mathbb{P}^{n-1}\)-functor [Add16] and the analogue holds for generalised Kummer varieties [Mea15] which yields non-standard autoequivalences on these two series of hyperkähler varieties.
The images of vector bundles under the FM transform $F'' : D(X) \to D(X^{[n]})$ are again vector bundles called *tautological bundles*. Not surprisingly, they play an important role for the geometry of Hilbert schemes of points on surfaces; see [Leh99; LS01; LS03; Sta16]. The composition $\Phi F'' : D(X) \to D_{\mathcal{E}_n}(X^{[n]})$ with the derived McKay equivalence was computed by Scala [Sca09a].

### 0.2 Summary of results

This thesis consists of 9 research papers of mine which I wrote between my PhD graduation in August of 2012 and January of 2017. Of these 9 papers, 4 are published in journals with peer-review and a further two are accepted for publication. The remaining 3 are all submitted to journals and available on the arXiv as well as on my personal website [https://sites.google.com/site/andkrugmath](https://sites.google.com/site/andkrugmath). Four of the 9 papers are written jointly with coauthors.

#### Papers constituting this thesis


Most of the results can be regarded as contribution towards the goals $P_1$, $P_2$, $P_3$, and $P_4$. A central theme is the McKay correspondence and the two series of varieties which are studied the most intensively are Hilbert schemes of points on surfaces and generalised Kummer varieties.

#### 0.2.1 Equivariant derived functors and a necessary condition for derived McKay equivalences [1]

The theory of Fourier–Mukai transforms can be generalised to *equivariant Fourier–Mukai transforms*. Let $G$ and $H$ be finite groups acting on varieties $X$ and $Y$, respectively. Then
every object $P \in D_{G\times H}(X \times Y)$ gives rise to an equivariant FM transform $D_G(X) \to D_H(Y)$. These equivariant FM transforms are ubiquitous when dealing with the derived McKay correspondence and applications. In particular, the derived McKay equivalence itself is an equivariant FM transform.

The first part of the article [1] is a theoretical treatment of the question when a ‘usual’ Fourier–Mukai transform $D(X) \to D(Y)$ lifts to an equivariant one. We give a criterion for such a lift to exists and show that the lift inherits a large number of possible properties of the original FM transform: being an equivalence, fully faithful, spherical, or a $P$-functor. This generalises results of the foundational paper [Plo07]. We are able to treat equivariant derived categories and equivariant FM transforms in the above sense (where $G$ is a finite subgroup of $\text{Aut}(X)$) in the same way as equivariant derived categories with respect to a finite subgroup $G \subset \text{Pic}(X)$ acting on $D(X)$ by tensor products. This allows us to deduce criteria for lift and descend of FM transforms along Galois covers which generalise results of [BM98] and [LP15, Sect. 2].

The second part of the article [1] gives applications of the first part in the context of the derived McKay correspondence. Probably the most interesting application is a criterion for a global quotient stack $[X/G]$ not to have a smooth projective variety as a FM partner. Recall that a crucial assumption for the derived McKay correspondence Theorem 0.1.4 is that $\omega_M$ is locally $G$-trivial. We prove the following partial converse

**Theorem 0.2.1** ([1, Proposition 1.5.6 & Proposition 1.5.7]). Let $M$ be a smooth projective variety together with a finite subgroup $G \subset \text{Aut}(M)$ such that $\omega_M$ is not locally $G$-trivial. Then there is no smooth projective variety $Y$ such that $D(Y) \cong D_G(X)$ if one of the following holds:

1. As a non-equivariant bundle, $\omega_M$ is trivial.
2. As a non-equivariant bundle, $\omega_M$ is ample or anti-ample (and a mild technical assumption holds).

Since the two cases in which the theorem holds are quite opposite, it makes sense to conjecture in general that $[M/G]$ cannot have a smooth projective variety as an FM partner as soon as $\omega_M$ is not locally $G$-trivial. As a corollary of Theorem 0.2.1, we see that symmetric quotient stacks of curves $[\mathbb{C}^n/\mathbb{S}_n]$ never have varieties as FM partners.

Further aspects of [1] which are worth being pointed out are:

- A very short proof, in the special case of smooth projective varieties, of Balmer’s result [Bal05] that the derived category $D(X)$ together with its tensor structure determines the scheme $X$; see [1, Subsection 1.3.5].

- There are two variants in the definition of spherical functors in the literature. Sometimes the equivalence $F^R \cong CF^L$ is required to be of a certain form. We discuss under which circumstances the a priori stronger condition follows as soon as there is any equivalence $F^R \cong CF^L$; see [1, Remark 1.3.20].

0.2.2 McKay correspondence for cyclic quotients and categorical crepant resolutions [2]

One of the simplest series of higher dimensional isolated quotient singularities is given by the cones $C_{m,n} \subset A^N$, where $N = \binom{n+m-1}{m}$, over the Veronese embeddings $\nu_m : \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{N-1}$. 

Indeed, we have $C_{m,n} \cong \mathbb{A}^n/\mu_m$ where the cyclic group $\mu_m$ acts on $\mathbb{A}^n$ by multiplication by a primitive $n$-th root of unit. The blow-up $\widetilde{Y} \to C_{m,n}$ in the origin is a resolution of the singularity and its derived category has been studied in [Abu16, Sect. 4], [IU15], and [Kaw16, Ex. 4].

We consider the following generalised set-up. Let $\mu_m$ act on a smooth variety $X$. We assume that only 1 and $\mu_m$ occur as the isotropy groups of the action and write $S := \text{Fix}(\mu_m) \subset X$ for the fixed point locus. Furthermore, we assume that $\mu_m$ acts diagonally with only one eigenvalue on the normal bundle $N_{S/X}$. Then the blow-up $\widetilde{Y} \to X/\mu_m$ in $S$ is a resolution of singularities. A particular case of this set-up, for $m = 2$, is the product $X = T^2$ of a smooth variety $T$. In this case, the resolution is given by the Hilbert scheme of two points: $\widetilde{Y} \cong T^{[2]}$.

First, we prove that the blow-up resolution equals the $\mu_m$-Hilbert scheme in our set-up: $\widetilde{Y} \cong \text{Hilb}^{\mu_m}(X)$; see [2, Proposition 1.1].

Setting $n := \text{codim}(S \hookrightarrow X)$ we obtain the following trichotomy which confirms the prediction of the DK hypothesis (see Subsection 0.1.5) in this case.

**Theorem 0.2.2 ([2, Theorem 2.1.2 & Theorem 2.4.1]).**

1. For $m > n$, there is a fully faithful embedding $\Phi : D(\widetilde{Y}) \hookrightarrow D_{\mu_m}(X)$ and a semi-orthogonal decomposition of $D_{\mu_m}(X)$ consisting of $\Phi(D(\widetilde{Y}))$ and $m - n$ pieces equivalent to $D(S)$.

2. For $n > m$, there is a fully faithful embedding $\Psi : D_{\mu_m}(X) \hookrightarrow D(\widetilde{Y})$ and a semi-orthogonal decomposition of $D(\widetilde{Y})$ consisting of $\Psi(D_{\mu_m}(X))$ and $n - m$ pieces equivalent to $D(S)$.

3. For $n = m$, we have $D(\widetilde{Y}) \cong D_G(X)$.

We study the case $m = n$ in further detail. The FM transforms $D(S) \to D_{\mu_m}(X)$ and $D(S) \to D(\widetilde{Y})$ which are fully faithful in the cases $m > n$ and $n > m$, respectively, become spherical functors for $m = n$. We are able to describe the tensor products induced by the equivalence $D(\widetilde{Y}) \cong D_G(X)$ on $D(\widetilde{Y})$ and $D_G(X)$ in terms of the twists along these spherical functors. Furthermore, we get a first example of a stacky 'twist-twist=flop' principle; see [2, Subsection 2.4.6]. In [2, Section 2.5], we introduce a general candidate for a categorical crepant resolution of a variety and test this candidate in our set-up of cyclic quotient singularities. In [2, Section 2.6], as an application of Theorem 0.2.2, we construct stability conditions on resolutions of Kummer threefolds.

Further aspects of [1] which are worth being pointed out are:

- We develop a theory of relative tilting objects in [2, Subsection 2.2.7]; compare [BB17b] for a similar approach.

### 0.2.3 Spherical functors on the Kummer surface [3]

Recall the discussion of Subsection 0.1.3: The Fourier–Mukai transform along the ideal sheaf of the universal family induces a $\mathbb{F}^{n-1}$-functor $F : D(X) \to D(X^{[n]})$ with cotwist $[-2]$ for $n \geq 2$; see [Add16]. The same holds for the analogue FM transform $F_K : D(A) \to D(K_n A)$ to the generalised Kummer surface for $n \geq 2$; see [Mea15]. For $n = 1$ the universal family of $X^{[1]} = X$ is the diagonal. Hence, $F = \text{FM}_{I_X} : D(X) \to D(X)$ is an autoequivalence, namely the spherical twist $T_{O_X}[1]$.  

25
The \( n = 1 \) case becomes much more interesting for the Kummer varieties where \( KA := K_1A \) is the Kummer K3 surface. This is what we study in [3].

**Theorem 0.2.3** ([3]). The FM transforms \( F, F'' : D(A) \rightarrow D(KA) \) along the ideal sheaf and the structure sheaf of the universal family are both split spherical functors with cotwist \((-1)^* \in \text{Aut}(D(A))\).

This seems very surprising in light of Bridgeland’s conjecture: For a K3 surface the group of derived autoequivalences should be generated by standard autoequivalences and twists along spherical objects (not functors); see Subsection 0.1.3. However, we are able to bring Theorem 0.2.3 in accordance with Bridgeland’s conjecture by also proving

**Theorem 0.2.4** ([3]). Both spherical twists \( T_F, T_{F''} \in \text{Aut}(D(KA)) \) are compositions of twists along the structure sheaves of the 16 exceptional \((-2)\)-curves (which are spherical objects) on the Kummer surface and tensor products by line bundles.

Further aspects of [3] which are worth being pointed out are:

- It is conjectured that every spherical object in the derived category of a smooth projective variety has a non-empty orthogonal complement. In contrast, the images of our spherical functors have an empty orthogonal complement; see [3, Corollary 3.2.6].

### 0.2.4 Varieties with \( \mathbb{P} \)-units [4]

As mentioned in Section 0.1.3, there are conjectured deep relations between \( \mathbb{P} \)-functors and hyperkähler varieties. The starting point of [4] is the observation that, for \( X \) a smooth projective variety of dimension \( 2n \), we have

\[
X \text{ is hyperkähler } \iff \mathcal{O}_X \in D(X) \text{ is a } \mathbb{P}^n\text{-object}.
\]

This follows quite immediately from [HN11, Prop. A.1] but we give a different proof in [4, Subsection 4.3.3]. Furthermore, almost by definition, a smooth projective variety \( X \) is a strict Calabi–Yau variety if and only if \( \mathcal{O}_X \in D(X) \) is a spherical object. One way to rephrase these two observations without referring to derived categories is: Let \( X \) be a variety with trivial canonical bundle. Then

\[
X \text{ is a strict Calabi–Yau of dimension } k \iff H^\ast(\mathcal{O}_X) \cong \mathbb{C}[x]/x^k, \quad \text{deg } x = k;
\]

\[
X \text{ is hyperkähler of dimension } 2n \iff H^\ast(\mathcal{O}_X) \cong \mathbb{C}[x]/x^{n+1}, \quad \text{deg } x = 2.
\]

In regard of this, it seems natural to consider, as a common generalisation of strict Calabi–Yau and hyperkähler varieties, the class of varieties with trivial canonical bundle and the property that the algebra \( H^\ast(\mathcal{O}_X) \) is generated by one element. If \( X \) is a variety with \( \omega_X \cong \mathcal{O}_X \) and \( H^\ast(\mathcal{O}_X) \cong \mathbb{C}[x]/x^{n+1} \) where \( \text{deg } x = k \), we say that \( X \) has a \( \mathbb{P}^n[k] \)-unit. Indeed, this is equivalent to the condition that \( \mathcal{O}_X \in D(X) \) is a \( \mathbb{P}^n[k] \)-object, which means that \( \text{FM}_{\mathcal{O}_X} : D(\text{pt}) \rightarrow D(X) \) is a \( \mathbb{P}^n \)-functor with cotwist \([-k]\). Using these notions, the classical \( \mathbb{P} \)-objects of Huybrechts and Thomas [HT06] become \( \mathbb{P}[2] \)-objects and hyperkähler varieties become varieties with \( \mathbb{P}[2] \)-units. Furthermore, strict Calabi–Yau varieties of dimension \( k \) become varieties with \( \mathbb{P}^1[k] \)-units.
This raises the basic question for which values of \( n > 2 \) and \( k > 2 \) varieties with \( \mathbb{P}^n[k] \)-units exist. The first simple observation is that \( k \) must be even since the cup product on \( H^*(\mathcal{O}_X) \) is graded commutative. We show that the question of existence of varieties with \( \mathbb{P}^n[k] \)-units for \( k \geq 4 \) is closely related to the question of existence of higher dimensional Enriques varieties. Such higher dimensional analogues of Enriques surfaces where studied by Boissière, Nieper-Wißkirchen, and Sarti [BNS11] and independently by Oguiso and Schröer [OS11]. They provide examples of Enriques varieties of dimension 4 and 6 as quotients of generalised Kummer varieties. We say that a variety is a strict Enriques variety if it is an Enriques variety in the sense of [BNS11] and in the sense of [OS11].

**Theorem 0.2.5** ([4, Theorem 4.1.3]). Let \( n + 1 = p' \) be a prime power. Then the following are equivalent:

(i) There exists a variety with \( \mathbb{P}^n[4] \)-unit,

(ii) There exists a variety with \( \mathbb{P}^n[k] \)-unit for every even \( k \),

(iii) There exists a strict Enriques variety of dimension \( 2n \).

For \( n + 1 \) arbitrary, the implications (iii)\( \Rightarrow \) (ii)\( \Rightarrow \) (i) are still true and we also prove a statement which is only slightly weaker than (i)\( \Rightarrow \) (iii); see [4, Theorem 4.5.8].

**Corollary 0.2.6.** For \( n = 2 \) and \( n = 3 \) there are examples of varieties with \( \mathbb{P}^n[k] \)-units for every even \( k \in \mathbb{N} \).

Our construction of varieties with \( \mathbb{P} \)-units out of Enriques varieties can be seen as a generalisation of a construction of higher dimensional Calabi–Yau varieties of Cynk and Hulek [CH07]; see [4, Remark 4.4.7]. Some further constructions are described in [4, Subsection 4.6.1&4.6.2&4.6.3]. It would be very interesting to construct varieties with \( \mathbb{P}^n[k] \)-units directly, maybe as moduli spaces of objects on Calabi–Yau categories and then employ Theorem 0.2.5 to obtain, possibly new, Enriques or even hyperkahler varieties; see [4, Subsection 4.6.7] for some more details on this speculation.

Further aspects of [3] which are worth being pointed out are:

- We prove that the class of strict Enriques varieties is stable under derived equivalence; see [4, Subsection 4.6.5].

### 0.2.5 Semi-orthogonal decompositions on Hilbert schemes of points on surfaces and induced autoequivalences [5]

We have already seen that, for \( X \) a K3 surface, the Fourier–Mukai transform along the universal ideal sheaf \( F : D(X) \to D(X[n]) \) has very interesting properties. Thus, it makes sense to study this functor also for other surfaces.

**Theorem 0.2.7** ([5, Theorem 5.1.2]). Let \( X \) be a smooth projective surface with \( p_g = q = 0 \). In other words, \( \mathcal{O}_X \in D(X) \) is an exceptional object. Then, the Fourier–Mukai transform \( F = FM_{\mathcal{I}_Z} : D(X) \to D(X[n]) \) is fully faithful for every \( n \geq 2 \).

In the case that \( X \) is an Enriques surface, the canonical bundle \( \omega_{X[n]} \) of the Hilbert scheme is of order two and the canonical cover \( \pi : \tilde{X}[n] \to X[n] \) is a strict Calabi–Yau variety. We
prove that \( \bar{F} := \pi^* F : D(X) \to D(\widehat{X^{[n]}}) \) is a spherical functor and that the induced twist \( T_{\bar{F}} \in \text{Aut}(D(\widehat{X^{[n]}})) \) descends to a non-standard autoequivalence of \( D(X^{[n]}) \); see [5, Theorem 5.1.1].

Note that, concerning Problems P1 and P2, all phenomena occurring on surfaces translate to the associated Hilbert schemes due to Ploog’s theorem Theorem 0.1.5. We prove an analogue concerning Problem P3.

**Theorem 0.2.8** ([5, Proposition 5.4.5 & Remark 5.4.6, 5.4.7, 5.4.8]). Every (full, strong) exceptional sequence of the derived category \( D(X) \) induces one on the derived category \( D(X^{[n]}) \) of the Hilbert scheme. Similarly, semi-orthogonal decompositions induce semi-orthogonal decompositions and tilting objects induce tilting objects.

In the case that \( X \) is an Enriques surface, the induced exceptional objects on \( X^{[n]} \) pull back to spherical objects on the Calabi–Yau cover \( \widehat{X^{[n]}} \). Again, the spherical twists descend to give further non-standard autoequivalences of \( D(X^{[n]}) \).

All the proofs of the above results crucially use the McKay equivalence \( \Phi : D(X^{[n]}) \cong D_{\mathbb{S}_n}(X^n) \). Motivated by this, we study the equivariant derived category \( D_{\mathbb{S}_n}(X^n) \) (or, equivalently, the derived category of the symmetric quotient stacks \( [X^n/\mathbb{S}_n] \)) for \( X \) a smooth projective variety of higher dimension. We show that a slight variant of \( \Phi F : D(X) \to D_{\mathbb{S}_n}(X^n) \), that we call truncated universal ideal functor, is still fully faithful for \( O_X \) exceptional and a \( \mathbb{P}^{n-1} \)-functor for \( O_X \) spherical (i.e. \( X \) a strict Calabi–Yau variety); see [5, Section 5.5].

Further aspects of [3] which are worth being pointed out are:

- Most known examples of phantom categories occur on surfaces of general type with \( p_g = q = 0 \); see Section 0.1.4. Hence, by Theorem 0.2.7 we get phantom categories on Hilbert schemes of points; see [5, Remark 5.3.3].

### 0.2.6 \( \mathbb{P} \)-functor versions of the Nakajima operators [6; 7]

So far, all the results we described concerning the derived category \( D(X^{[n]}) \) depend crucially on the properties of the derived category \( D(X) \) of the underlying surface. Furthermore, they all turn out to have analogous statements for the equivariant derived category \( D_{\mathbb{S}_n}(X^n) \) for \( X \) of arbitrary dimension. It is only that the surface case is the most interesting due to the derived McKay equivalence \( D(X^{[n]}) \cong D_{\mathbb{S}_n}(X^n) \) which gives interpretations in terms of an honest variety instead of an orbifold.

In [6; 7] we give a construction of functors \( H_{\ell,n} : D_{\mathbb{S}_\ell}(X \times X^\ell) \to D_{\mathbb{S}_{n+\ell}}(X^{n+\ell}) \) which can be regarded as lifts of the Nakajima operators to the level of the derived categories. The behaviour of these functors depends crucially on the dimension of \( X \) but not on any other properties of the derived category \( D(X) \).

First, we observe that the \( \mathbb{S}_\ell \times \mathbb{S}_n \)-invariant subvariety

\[
\hat{Z}^{n,\ell} = \{ (x, x_1, \ldots, x_\ell, y_1, \ldots, y_n) \mid n \cdot x + \sum_{i=1}^{\ell} x_i = \sum_{j=1}^{n} y_j \} \subset X \times X^\ell \times X^n
\]

(recall that we write unordered tuples of points as formal sums) is a natural analogue of the incidence scheme \( Z^{\ell,n} \subset X \times X^{\ell} \times X^{n+\ell} \) (see (4)) on the equivariant side of the McKay correspondence.

For \( \ell = 0 \), the equivariant version of the incidence subscheme is given by the small diagonal \( \hat{Z}^{0,n} = \Delta \subset X \times X^n \).
Theorem 0.2.9 ([6, Theorem 6.1.1]). For every smooth quasi-projective surface $X$ and $n \geq 2$, the functor

$$H := H_{0,n} := \text{FM}_{C_{\Delta}} = \delta_* : D(X) \to D_{\mathcal{E}_n}(X^n)$$

is a $\mathbb{P}^{n-1}$-functor with $\mathbb{P}$-cotwist $S_X^{-1} = (\_ ) \otimes \omega_X^\vee [-2]$.

For arbitrary $\ell, n \in \mathbb{N}$, we construct an equivariant complex $P_{\ell,n} \in D_{\mathcal{E}_{\ell} \times \mathcal{E}_n}(X \times X^\ell \times X^n)$ supported on $Z^{n,\ell}$ and prove

Theorem 0.2.10 ([7, Theorem C]). For every smooth quasi-projective surface $X$ and $n > \max\{\ell, 1\}$, the functor

$$H_{\ell,n} := \text{FM}_{P_{\ell,n}} : D_{\mathcal{E}_{\ell}}(X \times X^\ell) \to D_{\mathcal{E}_n}(X^n)$$

is a $\mathbb{P}^{n-1}$-functor with $\mathbb{P}$-cotwist $S_X^{-1} = (\_ ) \otimes \omega_X^\vee [-2]$.

This result can be seen as a partial categorification of the Heisenberg action on the cohomology of the Hilbert schemes. Indeed, for $\omega_X$ trivial, Theorem 0.2.10 says that

$$H_{\ell,n}^R H_{\ell,n} \cong \text{id} \oplus [-2] \oplus [-4] \oplus \cdots \oplus [-2(n-1)].$$

Since even shifts act trivially on the cohomology, this tells us that $H_{\ell,n}^R H_{\ell,n}$ acts on $H^*(X^\ell, \mathbb{Q})$ by $n \cdot \text{id}$, which should be compared with the formula for the Lie bracket (3) of the Heisenberg algebra; see [7, Subsection 7.1.3] for details on this.

The $\mathbb{P}$-functors $H_{\ell,n}$ induce very interesting autoequivalences of $D(X^m) \cong D_{\mathcal{E}_m}(X^m)$ where $m = n + \ell$. The image of the functor $H_{\ell,n}$ is supported on certain partial diagonals of $X^m$, which happen to be the partial fixed point loci under the $\mathcal{E}_m$-action. Hence, the associated $\mathbb{P}$-twists $P_{H_{\ell,n}}$ can be regarded as ‘characteristic functions’ of these partial fixed point loci: They shift skyscraper sheaves of orbits on these loci by a certain value while they fix skyscraper sheaves of orbits which lie outside of the fixed point locus; compare (2). These partial fixed point loci also have a natural interpretation in terms of the loci of $X^m$ parametrising length $n$ subschemes with given types of non-reducedness. We discuss the $\mathbb{P}$-twists $P_{H_{\ell,n}}$, the subgroup of $\text{Aut}(D(X^m))$ generated by them, and relations to other known autoequivalences in [7, Subsection 7.5.3–7.5.6]. We also conjecture that, for $\omega_X$ ample or anti-ample and low values of $n$, these $\mathbb{P}$-twists along with the standard autoequivalences generate the whole $\text{Aut}(D(X^m))$; see [7, Subsection 7.5.9] for more details on this and some further speculations.

Our construction does not seem to have very good properties if we replace the surface $X$ by a variety of higher dimension; see [7, Subsection 7.1.2]. However, it also gives something remarkable in the curve case.

Theorem 0.2.11 ([7, Proposition A & Corollary B & Subsection 7.5.7]). For every smooth quasi-projective curve $C$ and $n > \max\{\ell, 1\}$, the functor

$$H_{\ell,n} := \text{FM}_{P_{\ell,n}} : D_{\mathcal{E}_{\ell}}(C \times C^\ell) \to D_{\mathcal{E}_n}(C^n)$$

is fully faithful. We get an induced semi-orthogonal decomposition

$$D_{\mathcal{E}_m}(C^n) = (A_{0,m}, A_{1,m-1}, \ldots, A_{r,m-r}, \pi^*(D^b(C^{(n)})), ?)$$

where $A_{\ell,m-\ell} = H_{\ell,m-\ell}(D^b_{\mathcal{E}_{\ell}}(C \times C^\ell))$.  

29
In general, for a finite group $G$ acting on a variety $M$, the orbifold cohomology of the quotient orbifold $[M/G]$ is given by the direct sum

$$
H^*([M/G]) \cong \bigoplus_{g \in \text{conj}(G)} H^*(M^g)^{\text{C}(g)} \cong \bigoplus_{g \in \text{conj}(G)} H^*(M^g / \text{C}(g))
$$

(up to certain degree shifts of the direct summands) where \(\text{conj}(G)\) denotes the set of conjugacy classes and \(M^g\) is the subvariety of \(g\)-fixed points on which the centraliser \(\text{C}(g)\) acts. For \(G = \mathbb{G}_m, M = C^m\) and \(m = 2, 3\) this reduces to the following formulae for the cohomology of the symmetric quotient orbifolds

$$
H^*((C^2/\mathbb{G}_2)) \cong H^*(C^{(2)}) \oplus H^*(C), \quad H^*((C^3/\mathbb{G}_3)) \cong H^*(C^{(3)}) \oplus H^*(C \times C) \oplus H^*(C).
$$

One can show that for \(m = 2, 3\) the unknown factor \(\mathbb{G}\) of the semi-orthogonal decomposition (5) vanishes. Hence, (5) categorifies the direct sum decomposition of the orbifold cohomology; compare Subsection 0.1.7. For higher \(m\), the semi-orthogonal decomposition (5) only reflects some but not all the factors of the direct sum decomposition (6).

More recently, Polishchuk and Van den Bergh [PV15] constructed a full categorification of the decomposition of \(H^*(C^m/\mathbb{G}_m)\) for arbitrary \(m\) by very different methods using the Springer correspondence. They also make a general conjecture under which circumstances (6) admits a categorification by a semi-orthogonal decomposition of \(\mathcal{D}(M/G) = \mathcal{D}_G(M)\). Namely, this should be the case if \(G \subset \text{Aut}(M)\) is generated by pseudo-reflections, i.e. elements of \(\text{Aut}(M)\) whose fixed point locus is of codimension 1.

Definitely, there cannot be such a semi-orthogonal decomposition of \(\mathcal{D}(M/G)\) for arbitrary global quotient orbifolds \([M/G]\). For example, if \(X\) is an abelian or K3 surface, then \(\mathcal{D}(X^n/\mathbb{G}_n) \cong \mathcal{D}(X^n)\) does not admit any semi-orthogonal decompositions at all; compare Subsection 0.1.4. However, one would still like the stratification of \(M\) into partial fixed point loci to be reflected on the level of the derived categories in some way. Maybe there are in general ‘characteristic autoequivalences’ for the partial fixed point loci similar to the \(\mathbb{P}\)-twists in the case of \(\mathcal{D}(X^n/\mathbb{G}_n)\) for \(X\) a surface. At least, we show that such autoequivalences also exist on \(\mathcal{D}(C^m/\mathbb{G}_m)\) for \(C\) a curve. The construction is similar to our construction of autoequivalences on the Hilbert schemes of Enriques surfaces: Using the 2:1 cover of orbifolds \(p:\ [\mathbb{C}^m/\mathbb{A}_m] \to [\mathbb{C}^m/\mathbb{G}_m]\), the fully faithful functors \(H_{t,m-t}\colon \mathcal{D}(C \times [\mathbb{C}^t/\mathbb{G}_t]) \to \mathcal{D}([\mathbb{C}^m/\mathbb{A}_m])\) lift to spherical functors \(p^*H_{t,m-t}\colon \mathcal{D}(C \times [\mathbb{C}^t/\mathbb{G}_t]) \to \mathcal{D}([\mathbb{C}^m/\mathbb{A}_m])\) and the associated spherical twists descend to autoequivalences of \(\mathcal{D}([\mathbb{C}^m/\mathbb{G}_m])\) which can be regarded as characteristic functions on the fixed point loci; see [7, Subsection 7.5.8].

Further aspects of [6] and [7] which are worth being pointed out are:

- We prove basic results concerning the compatibility of \(\mathbb{P}\)-twists with other autoequivalences; see [6, Lemma 6.2.4].

- For all of the above results, we also prove analogues for generalised Kummer varieties and generalised Kummer stacks; see [6, Theorem 6.6.1], [7, Proposition A’ & Corollary B' & Theorem C'].

- We give examples of braid relations between twists along spherical functors on hyperkähler fourfolds; see [7, Subsection 7.5.6].
0.2.7 Categorification of the Heisenberg action [8]

The construction of [7] only gives a partial categorification of the Heisenberg action on the cohomology of the Hilbert schemes. Roughly, it only reflects the Lie bracket (3) for \( n = -m \) but not the vanishing Lie bracket for \( n \neq -m \). In [8] we construct a categorical action of the Heisenberg algebra on \( D_X := \oplus_{n \geq 0} D([X^n/\mathcal{O}_X]) \) for every smooth projective variety \( X \). Our construction generalises aspects of constructions of Khovanov [Kho14], who treats the purely representation theoretic case where \( X = \text{pt} \) is a point, and Cautis and Licata [CL12], who treat the case that \( X \) is a minimal resolution of a Kleinian singularity. The construction of [8], in contrast to that of [7], does not lift the Nakajima operators themselves but instead lifts another set of generators of the Heisenberg algebra, known as halves of the vertex operators, to the level of the derived category. In particular, we get an action of the Heisenberg algebra on the Grothendieck groups of the Hilbert schemes of points on surfaces; compare [SV13; FT11] for such an action (with other additional features) in the case that \( X \) is the affine plane.

0.2.8 Tautological bundles and the McKay correspondence in the reverse direction [9]

Besides the FM transform \( F := \text{FM}_{\mathcal{O}_Z} : D(X) \to D(X^{[n]}) \), which we already intensively studied in the articles summarised above, there is a second, equally canonical FM transform, namely \( F' = \text{FM}_{\mathcal{O}_Z} : D(X) \to D(X^{[n]}) \). Since \( \Xi \) is flat and of rank \( n \) over \( X^{[n]} \), the functor \( F' \) maps vector bundles \( E \in V \mathcal{B}(X) \) of rank \( r \) to vector bundles \( E^{[n]} := F'(E) \in V \mathcal{B}(X^{[n]}) \) of rank \( nr \), called tautological bundles. The image \( \Phi(E^{[n]}) \) of these tautological bundles under the derived McKay equivalence was computed by Scala [Sca09a]. It is given by an \( \mathfrak{S}_n \)-equivariant complex \( C_E^\bullet \) on \( X^n \) of length \( n \) whose degree zero term is

\[
C^0_E = \oplus_{i=1}^n \text{pr}_i^* E
\]

where \( \text{pr}_i : X^n \to X \) is the projection to the \( i \)-th factor and the direct sum is equipped with a natural \( \mathfrak{S}_n \)-linearisation permuting the summands. Scala’s description of \( \Phi(E^{[n]}) \) has been used in [Sca09a; Sca09b; Sca15; Kru14a; Kru14b; Mea15; MM15] in order to prove many interesting consequences. However, the proofs are often computationally involved, mainly due to the higher degree terms of the complex \( C_E^\bullet \).

The main observation exploited in [9] is that it has benefits to consider the derived McKay correspondence in the reverse direction

\[
\Psi := \text{FM}_{\mathcal{O}_Z}^{-1} = (-)^{\mathfrak{S}_n} \circ q_* \circ p^* : D_{\mathfrak{S}_n}(X^n) \to D(X^{[n]})
\]

instead. The functor \( \Psi \) is again an equivalence, but not the inverse of \( \Phi \); see [9, Proposition 9.2.9]. Our technical main result is that, if we replace \( \Phi \) by \( \Psi^{-1} \), the higher order terms of \( C_E^\bullet \) vanish and we get a similarly simple description for the images of wedge powers of tautological bundles associated to line bundles on the surface.

**Theorem 0.2.12** ([9, Theorem 9.1.1]).

1. For every coherent sheaf \( E \in \text{Coh} \ X \), we have \( \Psi(C_E^0) \cong E^{[n]} \).

2. For every line bundle \( L \in \text{Pic} \ X \) and \( 0 \leq k \leq n \), we have

\[
\Psi(W^k(L)) \cong \wedge^k L^{[n]} \quad \text{where} \quad W^k(L) = \bigoplus_{I \subset \{1, \ldots, n\}, |I| = k} \text{pr}_I^*(L^{[k]})
\]
Here, \( \text{pr}_I : X^n \to X^k \) is the projection to the \( I \)-factors and \( \mathcal{W}^k(L) \) carries a \( \mathfrak{S}_n \)-linearisation by permutation of the direct summands together with appropriate signs; see [9, Definition 9.3.4] for details.

Probably the most important application of this theorem is the computation of homological invariants of tautological bundles and their wedge powers; see [9, Section 9.4\&9.5\&9.6]. However, we will not go into the details of this application in this introduction.

Instead, we want to point out two other aspects of Theorem 0.2.12 which are relevant to the study of the derived category of the Hilbert schemes of points. First, part 1 allows to give significantly easier proofs of the main properties ([Add16] and Theorem 0.2.7) of the functor \( F = F_{\mathfrak{M}_L} : D(X) \to D(X^{[n]}) \) and explains the occurrence of the truncated universal ideal functors of [5, Section 5.5]; see [9, Remark 9.3.15]. Secondly, the \( \mathfrak{S}_n \)-equivariant bundles \( \mathcal{W}^k(L) \) play an important role in both, the construction of exceptional sequences (see Theorem 0.2.8) and the Heisenberg action on the derived category of the symmetric quotient stacks (see Subsection 0.2.7). Hence, part 2 of Theorem 0.2.12 can be regarded as a partial geometric interpretation of these features of the equivariant derived categories in terms of the derived categories of Hilbert schemes of points on surfaces; see [9, Remark 9.3.12].

Further aspects of [9] which are worth being pointed out are:

- Again, there are analogous results for generalised Kummer varieties; see [9, Remark 9.3.14 & 9.4.4].
- We also compute homological invariants of tautological bundles on Hilbert schemes of points on curves; see [9, Subsection 9.6.1].
- We show that a conjecture of Wang and Zhou [WZ14, Conj. 1 & Sect. 2.3] cannot hold in full generality [9, Remark 9.6.5 & Subsection 9.6.2] but give some additional evidence for a restricted version of the conjecture [9, Remark 9.6.5].

References


Chapter 1

Equivalences of equivariant derived categories


Abstract

We investigate conditions for a Fourier-Mukai transform between derived categories of coherent sheaves on smooth projective stacks endowed with actions by finite groups to lift to the associated equivariant derived categories. As an application we give a condition under which a global quotient stack cannot be derived equivalent to a variety. We also apply our techniques to generalised Kummer stacks and symmetric quotients.

1.1 Introduction

If $Z$ is a smooth projective variety and $\mathcal{D}^b(Z)$ its bounded derived category of coherent sheaves, it is interesting to understand the group of autoequivalences of $\mathcal{D}^b(Z)$ or to investigate when $\mathcal{D}^b(Z)$ is equivalent to $\mathcal{D}^b(Z')$ for some other variety $Z'$. If the latter holds, it is reasonable to try and study what happens to the equivalence if we pass from $Z$ and $Z'$ to varieties which are closely connected to them geometrically.

As an example, we can consider a smooth projective variety with torsion canonical bundle of order $n = \text{ord}(\omega_X)$ and its canonical cover $p_X : \tilde{X} \to X$, where $\tilde{X} = \text{Spec}(\mathcal{O}_X \oplus \omega_X \oplus \ldots \oplus \omega_X^{n-1})$; see [BM98] or Section 1.4 for details. There is a free action of $G = \mathbb{Z}/n\mathbb{Z} = \langle \tau_X \rangle$ on $\tilde{X}$ such that $\tilde{X}/G = X$. If $Y$ is a second smooth projective variety with torsion canonical bundle of the same order, Bridgeland and Maciocia, see [BM98], show the following theorem:

Any equivalence $\tilde{F} : \mathcal{D}^b(X) \cong \mathcal{D}^b(Y)$ lifts to an equivalence $\mathcal{D}^b(\tilde{X}) \cong \mathcal{D}^b(\tilde{Y})$. Conversely, if $\tilde{F} : \mathcal{D}^b(\tilde{X}) \cong \mathcal{D}^b(\tilde{Y})$ is an equivariant equivalence, that is, there exists an integer $k$ coprime to $n$ such that $\tau_X^n \circ \tilde{F} \cong \tilde{F} \circ \tau_X^k$, then $\tilde{F}$ descends.

In [LP15], there is a similar statement where the canonical bundle is replaced by a torsion bundle $L \in \text{Pic}^0(X)$.

In this paper we generalise the above results by using a strictly equivariant approach to lift and descent. In Section 1.3 we first forget about the geometric situation of coverings and give criteria for an equivalence between the derived categories of smooth projective stacks associated to normal projective varieties with only quotient singularities (compare [Kaw04]).
to lift to an equivalence between the categories linearised by certain finite subgroups of the groups of standard autoequivalences, see Theorem 1.3.22. These results are achieved by a straightforward generalisation of the theory of equivariant Fourier-Mukai transforms as developed in [Plo07] and they translate to the following statement (which combines Propositions 1.4.2 and 1.4.3 with Remark 1.4.4) for descent and lift along quotients of smooth projective stacks by group actions generalising the criteria of [BM98] and [LP15]. See Definitions 1.3.4, 1.3.5, 1.3.8 and Section 1.4 for the notions used.

**Theorem 1.1.1.** Let \( \pi: \tilde{X} \to X \) and \( \pi: \tilde{Y} \to Y \) be Galois covers of smooth projective stacks with groups of deck transformations \( H \) and \( H' \), respectively. Let \( \Phi = \text{FM}_P: \text{D}^b(\tilde{X}) \to \text{D}^b(\tilde{Y}) \) be a Fourier-Mukai transform. If \( P \in \text{D}^b(\tilde{X} \times \tilde{Y}) \) is \( \mu \)-linearisable for some isomorphism \( \mu: H \cong H' \), then there is a \( \mu \)-equivariant descent \( \Psi: \text{D}^b(X) \to \text{D}^b(Y) \). In addition, if \( \Phi \) is an equivalence or fully faithful, so is \( \Psi \). If \( \Phi \) is spherical or \( \mathbb{P}^n \) satisfying a technical assumption, then the same holds for \( \Psi \). If \( H \) and \( H' \) are abelian, there is an analogous criterion for lift of Fourier-Mukai transforms along the Galois covers. If \( H \) and \( H' \) are cyclic, the condition that \( P \) is \( \mu \)-linearisable can be replaced by the condition that \( \Phi \) is \( \mu \)-equivariant.

The paper is organised as follows. In Section 2 we collect background material on group actions, equivariant categories and Fourier-Mukai transforms. Section 3 is devoted to the study of lifts of FM transforms to equivariant categories. The achieved results are interpreted geometrically in Section 4. In Section 1.5 we give more specific applications. For instance, we describe conditions under which a global quotient stack cannot be derived equivalent to a smooth variety; see Proposition 1.5.6. Furthermore, we study the derived category of symmetric quotients and closely related stacks. In the appendix we investigate necessary conditions for lifts.

**Conventions.** Varieties or stacks are assumed to be smooth and projective unless stated otherwise and to be defined over the complex numbers. All functors between derived categories are assumed to be derived although this will not be reflected in the notation.

**Acknowledgements.** We thank David Ploog for his comments and the referee for many helpful suggestions which improved the manuscript. A. K. was financially supported by the research grant KR 4541/1-1 of the DFG and P. S. was partially financially supported by the RTG 1670 of the DFG.

### 1.2 Preliminaries

#### 1.2.1 Group actions on categories

Let \( G \) be a finite group and \( \mathcal{T} \) a category. A **categorical action** of \( G \) on \( \mathcal{T} \) is the data of an autoequivalence \( g^*: \mathcal{T} \to \mathcal{T} \) for every \( g \in G \) together with isomorphisms \( g^*h^* \cong (hg)^* \) satisfying a natural cocycle condition (see [Del97] or [Sos12] for details). Given such an action, a \( G \)-linearised (or \( G \)-equivariant) object is an object \( E \in \mathcal{T} \) together with a collection of isomorphisms \( \lambda_g: E \to g^*E \) for \( g \in G \) such that \( \lambda_e = \text{id}_E \) and such that the composition

\[
E \xrightarrow{\lambda_g} g^*E \xrightarrow{g^*\lambda_h} g^*h^*E \cong (hg)^*E
\]

equals \( \lambda_{hg} \) for every pair \( g, h \in G \). Given two \( G \)-linearised objects \((E, \lambda)\) and \((F, \mu)\) we have a \( G \)-action on \( \text{Hom}_\mathcal{T}(E, F) \) given by \( \varphi \cdot g := \mu^{-1}_g \circ g^*(\varphi) \circ \lambda_g \). We define the morphisms in
the category of $G$-linearised objects $T^G$, sometimes also denoted by $T_G$, to be the invariants under this action, i.e.

$$
\text{Hom}_{T^G}((E, \lambda), (F, \mu)) := \text{Hom}(E, F)^G
$$

(1.1)

consists of morphisms $\varphi : E \to F$ commuting with the linearisations. There exist natural functors between the category $T$ and the equivariant category $T^G$. Dropping the linearisations gives the restriction (also called the forgetful) functor $\text{Res} : T_G \to T$, $(E, \lambda) \mapsto E$.

Let $T = A$ be an additive (abelian) category and let the autoequivalences $g^*$ all be additive (exact). Then $A^G$ is again an additive (abelian) category (see [Sos12, Prop. 3.2]). In this case there exists the inflation functor $\text{Inf} : A \to A^G$ defined as follows: For $E \in A$ we have

$$
\text{Inf}(E) = \bigoplus_{g \in G} g^* E
$$

equipped with the linearisation given by permuting the direct summands. If $G' \subset G$ is a subgroup, there is also the partial restriction functor $\text{Res}_{G'}^G : T_G \to T_{G'}^G$ as well as a relative version of the inflation functor, namely

$$
\text{Inf}_{G'}^G : A_{G'}^G \to A^G, \quad E \mapsto \bigoplus_{[g] \in G/G'} g^* E
$$

and the linearisation can be described explicitly; see [Plo07].

**Lemma 1.2.1.** The functor $\text{Inf}_{G'}^G$ is left and right adjoint to $\text{Res}_{G'}^G$.

**Proof.** See [Ela14, Lem. 3.7] for the case that $G' = 1$. The general case is similar. \hfill \Box

Let a finite group $G$ act on a $\mathbb{C}$-linear category $T$ via $\mathbb{C}$-linear autoequivalences. Then the group of characters $\hat{G} = \text{Hom}(G, \mathbb{C}^*)$ acts on $T^G$ by tensor product. This means that a $G$-equivariant object $(E, \lambda)$ is sent to $(E, \lambda) \otimes \chi := (E, \lambda')$ with $\lambda'_g = \chi(g) \cdot \lambda_g$. The action on morphisms is trivial.

Every object $\text{Inf} E = \bigoplus_{g \in G} g^* E$ in the image of the inflation functor carries a $\hat{G}$-linearisation $\psi$ given by $\psi_\chi = \bigoplus_{g \in G} \chi(g)^{-1} \cdot \text{id}_{g^* E}$. This gives a functor

$$
\text{Inf}_{U}^{G+} : T \to (T^G)^{\hat{G}}, \quad E \mapsto \text{Inf}^+(E) := (\text{Inf} E, \psi).
$$

For the proof of the following duality see [Ela14]. Recall that a category is called Karoubian if all idempotents have kernels.

**Proposition 1.2.2.** If $G$ is a finite abelian group acting on a Karoubian $\mathbb{C}$-linear category $T$, then $\text{Inf}^+ : T \to (T^G)^{\hat{G}}$ is an equivalence. \hfill \Box

We say that an object $E \in T$ is linearisable if it admits a linearisation. There is also the weaker notion of a $G$-invariant object $E \in T$. This means that there are isomorphisms $E \cong g^* E$ without requiring any condition on their compositions. The relationship of the two notions for simple objects is as follows.
Lemma 1.2.3 ([Plo07]). Let $G$ act on a $\mathbb{C}$-linear category $T$ by $\mathbb{C}$-linear automorphisms and let $E \in T$ be a $G$-invariant simple object, i.e. $\text{Hom}(E,E) = \mathbb{C}$. Then there exists a group cohomology class $[E] \in \text{H}^2(G, \mathbb{C}^*)$ with the property that $E$ is linearisable if and only if $[E] = 0$. Furthermore, if $[E] = 0$, then the set of isomorphism classes of $G$-linearisations of $E$ is a free $G$-orbit under the $G$-action described above.

Proof. This result is stated and proved in [Plo07, Lem. 1] in the special case that $T = \mathcal{D}^b(X)$ for a smooth projective variety $X$ and $G \subset \text{Aut}(X)$. The proof of the general result is exactly the same.

Remark 1.2.4. If $G = \mathbb{Z}/n\mathbb{Z}$ is cyclic, then $\text{H}^2(G, \mathbb{C}^*) = 0$. Thus, in this case every $G$-invariant simple object is automatically linearisable.

Remark 1.2.5. If $A$ is a symmetric monoidal category (see [Mac98, Ch. VII]), that is, there is a bifunctor $\otimes$ satisfying the usual properties, and if the group $G$ acts by monoidal equivalences, then $A^G$ inherits a monoidal structure. Indeed, if $(A, \lambda)$ and $(B, \mu)$ are linearised objects, then $\lambda \otimes \mu$ gives a linearisation of $A \otimes B$, since $\lambda_g \otimes \mu_g : A \otimes B \cong g^*(A) \otimes g^*(B) \cong g^*(A \otimes B)$. If, in addition, there is a unit with respect to $\otimes$ and $g^*$ sends the unit to itself for all $g \in G$, then $A^G$ also inherits a unit.

Remark 1.2.6. If $A$ is abelian with enough injective objects, a categorical action of $G$ on $A$ induces a categorical action of $G$ on $\mathcal{D}^b(A)$. These actions are compatible in the sense that the categories $\mathcal{D}^b(A^G)$ and $\mathcal{D}^b(A)^G$ are equivalent; see [Che14] or [Ela14]. Note that if $G$ acts on a triangulated category $T$, there is a priori no reason for $T^G$ to be a triangulated category since cones are not functorial and there is no canonical way to define a linearisation of a cone of a morphism between equivariant objects in $T$; see [Sos12] and [Ela14] for an approach to circumvent this problem using differential graded enhancements.

1.2.2 Smooth projective stacks and Fourier-Mukai transforms

The spaces we will work with are smooth projective stacks satisfying the assumptions of [Kaw04]. This means that whenever we speak of a smooth projective stack in the following, we mean a stack $\mathcal{X}$ which is naturally associated with a normal projective variety with only quotient singularities; see [Kaw04, Sect. 4] for details.

Let $\mathcal{X}$ and $\mathcal{Y}$ be smooth projective stacks. For any object $P \in \mathcal{D}^b(\mathcal{X} \times \mathcal{Y})$ there is the associated Fourier-Mukai transform

$$\text{FM}_P := \text{pr}_{\mathcal{Y}*}(\text{pr}_{\mathcal{X}}^*(\_ \otimes P) : \mathcal{D}^b(\mathcal{X}) \to \mathcal{D}^b(\mathcal{Y}).$$

Let $\mathcal{Z}$ be a third smooth projective stack and $Q \in \mathcal{D}^b(\mathcal{Y} \times \mathcal{Z})$. Then $\text{FM}_Q \circ \text{FM}_P \cong \text{FM}_{Q \circ P}$, where

$$Q \circ P := \text{pr}_{\mathcal{X} \times \mathcal{Z}}(\text{pr}_{\mathcal{Y} \times \mathcal{Z}}^*(Q) \otimes \text{pr}_{\mathcal{X} \times \mathcal{Y}}^* P)$$

is the convolution product.

Note that the kernel $P$ also induces a second Fourier-Mukai transform $\mathcal{D}^b(\mathcal{Y}) \to \mathcal{D}^b(\mathcal{X})$ in the opposite direction. To avoid the risk of confusing the two FM transforms we will sometimes write them as $\text{FM}_P^{\mathcal{X} \to \mathcal{Y}} : \mathcal{D}^b(\mathcal{X}) \to \mathcal{D}^b(\mathcal{Y})$ and $\text{FM}_P^{\mathcal{Y} \to \mathcal{X}} : \mathcal{D}^b(\mathcal{Y}) \to \mathcal{D}^b(\mathcal{X})$, respectively.
Lemma 1.2.7. Let $X_1$, $X_2$, $Y_1$, and $Y_2$ be smooth projective stacks, and let $R \in \text{D}^b(X_1 \times X_2)$ and $P_i \in \text{D}^b(X_i \times Y_i)$ for $i = 1, 2$ be arbitrary. We consider the exterior tensor product $P_1 \boxtimes P_2 \in \text{D}^b((X_1 \times X_2) \times (Y_1 \times Y_2))$ and set $S := \text{FM}_{P_1 \boxtimes P_2}(R) \in \text{D}^b(Y_1 \times Y_2)$. Then
\[
\text{FM}_{S}^{Y_1 \rightarrow Y_2} \cong \text{FM}_{P_2}^{X_2 \rightarrow Y_2} \circ \text{FM}_{R}^{X_1 \rightarrow X_2} \circ \text{FM}_{P_1}^{X_1 \rightarrow Y_1}.
\]

Proof. For varieties this is [Orl03, Prop. 2.1.6]. The proof in the case of stacks is exactly the same. 

Remark 1.2.8. Any stack in our sense is actually a global quotient stack of the form $[X/G]$, where $X$ is a quasi-projective scheme and $G$ a reductive algebraic group acting linearly; see, for example, [Kre09, Prop. 5.1]. Note that sheaves on such a quotient stack are the $G$-equivariant sheaves on $X$. However, since $G$ is not necessarily finite, the condition to be equivariant has to be reformulated, see, for instance, [HL10, Subsect. 4.2]. 

In all of our applications, the stacks will even be global finite quotient stacks, that is, smooth projective stacks of the form $X = [X/G]$ for $X$ a smooth projective variety and $G \subset \text{Aut}(X)$ a finite subgroup. In this case there are equivalences $\text{Coh}(X) \cong \text{Coh}_G(X)$ and, accordingly, $\text{D}^b(X) \cong \text{D}^b_2(X) := \text{D}^b(\text{Coh}_G(X))$. Under these equivalences, the Fourier-Mukai transforms are given by the equivariant Fourier-Mukai transforms of [Plo07].

1.3 Lifts of equivalences to equivariant categories

1.3.1 Linearisations by standard autoequivalences

Let $X$ be a smooth projective stack. Every automorphism $\varphi \in \text{Aut}(X)$ (note that this is really a group in our setting and not a possibly more complicated categorical object as for general stacks) gives the pull-back autoequivalence $\varphi^* : \text{Coh}(X) \rightarrow \text{Coh}(X)$. Furthermore, every line bundle $L \in \text{Pic}(X)$ induces the autoequivalence $M_L := (\_ \otimes L) : \text{Coh}(X) \rightarrow \text{Coh}(X)$.

We set $A(X) := \text{Pic}(X) \rtimes \text{Aut}(X)$. The product structure in $A(X)$ is given by $(L, \varphi) \cdot (\mathcal{K}, \psi) = (\psi^* L \otimes \mathcal{K}, \varphi \circ \psi)$.

Definition 1.3.1. There is an automorphism $c$ of $A(X)$ defined as follows. If $h = (L, \varphi) \in A(X)$, then $c(h) := \tilde{h} := (L^{-1}, \varphi)$. We say that a subgroup $H \subset A(X)$ is $c$-invariant if $\tilde{H} = H$. Furthermore, for $h = (L, \varphi)$, we set $p(h) := (\mathcal{O}, \varphi)$.

Remark 1.3.2. For $L \in \text{Pic}(X)$ the FM kernel of $M_L$ is given by the pushforward along the diagonal $\Delta, L \in \text{D}^b(X \times X)$ and for $\varphi \in \text{Aut}(X)$ the FM kernel of the pushforward $\varphi_*$ is the structure sheaf of the graph $\mathcal{O}_{\Gamma_{\varphi}} \in \text{D}^b(X \times X)$. Note that the FM transform in the opposite direction $\text{FM}_{X \leftarrow Y}$ is again $M_L$, while $\text{FM}_{\mathcal{O}_{\Gamma_{\varphi}}}$ is given by $\varphi^*$, the inverse of $\varphi_*$. This different behaviour can be seen as the reason for the occurrence of the automorphism $c$.

In the following we will consider finite $c$-invariant subgroups $H \subset A(X)$ and assume that they act categorically on $\text{Coh}(X)$. With this we mean that there is a categorical action of $H$ on $\text{Coh}(X)$ such that $h^* = M_L \circ \varphi^*$ for $h = (L, \varphi) \in H$. Clearly, any $H \subset A(X)$ acts categorically, since the isomorphisms $\varphi^* \circ \psi^* = (\psi \circ \varphi)^*$ satisfy cocycle conditions. One can also check that every $H \subset \text{Pic}(X)$ acts categorically; compare [Ela14] and note that the proof given there works for line bundles on smooth projective stacks as well.
In the applications we will always have either \( H \subset \text{Aut}(\mathcal{X}) \) or \( H \subset \text{Pic}(\mathcal{X}) \). The main reason that we nevertheless choose to work with general \( \psi \)-invariant subgroups acting categorically is to avoid dealing with two cases.

Given a \( \psi \)-invariant subgroup \( H \subset A(\mathcal{X}) \) acting categorically, we denote by \( \text{Coh}_H(\mathcal{X}) := \text{Coh}(\mathcal{X})^H \) the \( H \)-equivariant category. Its objects are sheaves \( E \in \text{Coh}(\mathcal{X}) \) together with a linearisation \( \lambda \) consisting of isomorphisms \( \lambda_h : E \xrightarrow{\sim} h^*E = \mathcal{L} \otimes \varphi^*E \) for \( h = (\mathcal{L}, \varphi) \in H \).

### 1.3.2 Fourier-Mukai transforms of \( \mu \)-type

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be smooth projective stacks and \( H \subset A(\mathcal{X}) \) and \( H' \subset A(\mathcal{Y}) \) be finite subgroups. There is an induced \( H \times H' \)-action on \( D^b(\mathcal{X} \times \mathcal{Y}) \) given for \( h \in H \) and \( h' \in H' \) by \((h \times h')^* := M_{\mathcal{L}_E \mathcal{L}'}, \circ(\varphi \times \varphi')^*\).

**Definition 1.3.3.** A group isomorphism \( \mu : H \to H' \) is called a \( \psi \)-isomorphism if \( c \circ \mu = \mu \circ c \). Note that if \( H \subset \text{Aut}(\mathcal{X}) \) and \( H' \subset \text{Aut}(\mathcal{Y}) \) or \( H \subset \text{Pic}(\mathcal{X}) \) and \( H' \subset \text{Pic}(\mathcal{Y}) \), then every isomorphism \( \mu : H \to H' \) is a \( \psi \)-isomorphism.

Given such a \( \psi \)-isomorphism, we let \( h \in H \) act on \( \text{Coh}(\mathcal{X} \times \mathcal{Y}) \) by \((h \times \mu(h))^* \). We denote the category of sheaves linearised with respect to this \( H \)-action by \( \text{Coh}_\mu(\mathcal{X} \times \mathcal{Y}) \) and its derived category by \( D^b_\mu(\mathcal{X} \times \mathcal{Y}) \). The objects in these categories are called \( \mu \)-linearised.

For \( \mathcal{P} = (P, \nu) \in D^b_\mu(\mathcal{X} \times \mathcal{Y}) \) there is an associated **Fourier-Mukai transform of \( \mu \)-type** given by

\[
\text{FM}_\mu^\mu := \text{pr}_{Y*} (\text{pr}_{X*}(\_) \otimes \mathcal{P}) : D^b(\mathcal{X}) \to D^b(\mathcal{Y}).
\]

For \( \mathcal{E} = (E, \lambda) \in D^b(\mathcal{X}) \), and \( h' \in H' \) there are the isomorphisms

\[
\alpha_{h'} := \text{pr}_{X*} \lambda_{\mu^{-1}(h')} \otimes \nu_{\mu^{-1}(h')} : \text{pr}_{X*}^* E \otimes P \xrightarrow{\sim} (p \circ \mu^{-1}(h') \times h')^*(\text{pr}_{X*}^* E \otimes P).
\]

Pushing forward by \( \text{pr}_{Y*} \) gives the \( H' \)-linearisation of \( \text{pr}_{Y*}(\mathcal{P} \otimes \text{pr}_{X*}^* \mathcal{E}) = \text{FM}_\mu^\mu(\mathcal{E}) \).

The sheaf \( O_{\Delta_X} \in D^b(\mathcal{X} \times \mathcal{X}) \) has a canonical \( \text{id}_H \)-linearisation so that the identity \( \text{id} : D^b_H(\mathcal{X}) \to D^b_H(\mathcal{X}) \) is a Fourier-Mukai transform of \( \text{id}_H \)-type. We denote \( O_{\Delta_X} \) equipped with this linearisation by \( O_{\Delta_X}^{\text{an}} \). Here, one can see the need to let \( h \) act by \((h \times h)^* \) and not by \((h \times h)^* \). More generally, for a character \( \varrho \in \hat{H} \), the tensor product \( M_{\varrho} : D^b_H(\mathcal{X}) \to D^b_H(\mathcal{X}) \) is the FM transform of \( \text{id}_H \)-type with kernel \( O_{\Delta_X}^{\text{an}} \otimes \varrho \).

Let \( \mu' : H' \to H'' \) be another \( \psi \)-isomorphism with \( H'' \subset A(\mathcal{Z}) \) for a third smooth projective stack \( \mathcal{Z} \). Then for \( \mathcal{P}' \in D^b(\mathcal{Y} \times \mathcal{Z}) \) the composition \( \text{FM}_\mu^\mu \circ \text{FM}_\mu^\mu \) is a Fourier-Mukai transform of \( \mu' \circ \mu \)-type with kernel given by the convolution product

\[
\mathcal{P}' \ast \mathcal{P} = \text{pr}_{\mathcal{X} \mathcal{Z}*} (\text{pr}_{\mathcal{X} \mathcal{Y}*} \mathcal{P} \otimes \text{pr}_{\mathcal{Y} \mathcal{Z}*} \mathcal{P}') \in D^b_{\mu' \circ \mu}(\mathcal{X} \times \mathcal{Z}). \tag{1.2}
\]

### 1.3.3 Equivariance of functors vs. equivariance of kernels

**Definition 1.3.4.** Let \( \mu : H \to H' \) be a \( \psi \)-isomorphism and \( \mathcal{P} \in D^b(\mathcal{X} \times \mathcal{Y}) \). We say that \( \mathcal{P} \) is \( \mu \)-linearisable if it admits a \( \mu \)-linearisation, i.e. if there is an object \( \mathcal{P} = (P, \nu) \in D^b(\mathcal{X} \times \mathcal{Y}) \) such that \( \text{Res} \mathcal{P} = \mathcal{P} \). There is also the weaker notion of \( \mu \)-invariance of \( \mathcal{P} \) which means that there are for \( h \in H \) isomorphisms \( \mathcal{P} \cong (h \times \mu(h))^* \mathcal{P} \) not necessarily satisfying the cocycle condition.

**Definition 1.3.5.** We say that an exact functor \( F : D^b(\mathcal{X}) \to D^b(\mathcal{Y}) \) is **\( \mu \)-equivariant** if \( F \circ h^* \cong \mu(h)^* \circ F \) for all \( h \in H \).
Then Denition 1.3.7.

Equivariant lifts

Lemma 1.3.6. Let $P \in \mathbb{D}^b(\mathcal{X} \times \mathcal{Y})$ and $F = \text{FM}_P : \mathbb{D}^b(\mathcal{X}) \to \mathbb{D}^b(\mathcal{Y})$.

1. $P$ is $\mu$-linearisable $\implies$ $P$ is $\mu$-invariant.

2. If $P$ is simple and $H$ is cyclic, the converse of (i) holds.

3. $P$ is $\mu$-invariant $\implies$ $F$ is $\mu$-equivariant.

4. If $F$ satisfies condition (1.3), the converse of (iii) holds.

Proof. The first assertion follows directly by definition and (ii) is Remark 1.2.4. For (iii) we use Lemma 1.2.7. It asserts in our situation that

$$\mu(h)^* \circ F \circ (h^*)^{-1} \simeq \text{FM}_{(\check{h} \times \mu(h))^*} P \quad \text{for all } h \in H;$$

compare Remark 1.3.2. Thus, the existence of an isomorphism $P \simeq (\check{h} \times \mu(h))^* P$ implies the $\mu$-equivariance of $F$. Finally, if $F$ satisfies condition (1.3), then $\mu(h)^* \circ F \circ (h^*)^{-1}$ does too. Thus, (iv) follows from (1.4) together with [CS07, Rem. 4.1].

1.3.4 Equivariant lifts

Definition 1.3.7. Let $F : \mathbb{D}^b(\mathcal{X}) \to \mathbb{D}^b(\mathcal{Y})$ and $\bar{F} : \mathbb{D}^b_{\check{H}}(\mathcal{X}) \to \mathbb{D}^b_{\check{H}}(\mathcal{Y})$ be exact functors. Then $\bar{F}$ is called a lift of $F$ if the following two conditions hold:

$$F \circ \text{Res} \simeq \text{Res} \circ \bar{F} : \mathbb{D}^b_{\check{H}}(\mathcal{X}) \to \mathbb{D}^b_{\check{H}}(\mathcal{Y}),$$

$$\text{Inf} \circ F \simeq \bar{F} \circ \text{Inf} : \mathbb{D}^b(\mathcal{X}) \to \mathbb{D}^b_{\check{H}}(\mathcal{Y}).$$

Definition 1.3.8. Let $\beta : \check{H}' \to \check{H}$ be an isomorphism of groups. A lift $\bar{F} : \mathbb{D}^b_{\check{H}}(\mathcal{X}) \to \mathbb{D}^b_{\check{H}}(\mathcal{Y})$ of $F : \mathbb{D}^b(\mathcal{X}) \to \mathbb{D}^b(\mathcal{Y})$ is called $\beta$-equivariant if $\bar{F} \circ M_{\beta(\check{\varrho})} \simeq M_{\check{\varrho}} \circ \bar{F}$ for all $\check{\varrho} \in \check{H}'$. Here, $M_{\check{\varrho}}$ denotes the tensor product by the character in the equivariant category.

The next lemma asserts that $F = \text{FM}_P$ lifts as soon as its kernel $P$ is $\mu$-linearisable for some $\mu$.

Lemma 1.3.9. Let $\mathcal{P} = (P, \nu) \in \mathbb{D}^b(\mathcal{X} \times \mathcal{Y})$ for some isomorphism $\mu : H \to \check{H}'$. Then

$$\bar{F} = \text{FM}^\mu_{\check{H}} : \mathbb{D}^b(\mathcal{X}) \to \mathbb{D}^b_{\check{H}}(\mathcal{Y})$$

is a $\check{\mu}$-equivariant lift of $F = \text{FM}_P : \mathbb{D}^b(\mathcal{X}) \to \mathbb{D}^b(\mathcal{Y})$.

Proof. The $\check{\mu}$-equivariance of $\bar{F}$ as well as the compatibility $F \circ \text{Res} \simeq \text{Res} \circ \bar{F}$ follow immediately from the definition of the FM transform of $\mu$-type.

For $E \in \mathbb{D}^b(\mathcal{X})$, we have $\bar{F} \text{Inf}(E) \cong \oplus_{h \in H} F h^*(E)$. By Lemma 1.3.6, this is isomorphic to $\oplus_{h' \in H} \check{h}^* F(E)$. Since the $H'$-linearisation of $\bar{F} \text{Inf}(E)$ is given by permutation of the direct summands, we get $\bar{F} \text{Inf}(E) \cong \text{Inf} F(E)$.

Our next result is a generalisation of [Plo07, Lem. 5].
Proposition 1.3.10. Let $P = (P, \nu) \in \mathcal{D}_\mu^b(\mathcal{X} \times \mathcal{Y})$ for some isomorphism $\mu: H \to H'$, $F = FM_P$, and $\tilde{F} = FM^\mu_P$. Then:

$$F \text{ is an equivalence } \implies \tilde{F} \text{ is an equivalence.}$$

Proof. The right adjoint kernel $P^R = P^\vee \otimes \text{pr}^*_\mathcal{X} \omega_{\mathcal{X}}[\dim \mathcal{X}] \in \mathcal{D}^b(\mathcal{Y} \times \mathcal{X})$ has an induced $\mu^{-1}$-linearisation $\tilde{\nu}$ given by $\tilde{\nu}_{\mu(h)} = (\nu_h^\vee)^{-1} \otimes \text{pr}^*_\mathcal{X} \lambda_p(h)[\dim \mathcal{X}]$ where $\lambda$ is the natural $p(H)$-linearisation of the canonical bundle given by the pushforward of $\dim \mathcal{X}$-forms. We denote $P^R$ equipped with this linearisation by $Q \in \mathcal{D}^b_{\mu^{-1}}(\mathcal{Y} \times \mathcal{X})$. The convolution product (1.2) is compatible with the restriction functor. So if $F$ is an equivalence, we have $\text{Res}(Q \ast P) = P^R \ast P = O_{\Delta Y}$. Furthermore, $\text{Hom}(O_{\Delta Y}, O_{\Delta Y}) \cong \Gamma(\mathcal{X}, O_{\mathcal{X}}) = \mathbb{C}$, which means that $O_{\Delta Y}$ is simple. Thus, $Q \ast P = O_{\Delta Y} \otimes \rho$ for some $\rho \in \mathcal{H}$ by Lemma 1.2.3. Hence, $\text{FM}^{\mu^{-1}}_{Q} \circ \tilde{F} = M_\rho$ is an equivalence. Similarly, we get that $\tilde{F} \circ \text{FM}^{\mu^{-1}}_{Q}$ is an equivalence, too. In summary, $\tilde{F}$ has a left and a right quasi-inverse functor. It follows that $\tilde{F}$ is an equivalence. $\square$

1.3.5 Monoidal and $\mathcal{X}$-linear autoequivalences

The reason that the calculus of Fourier-Mukai transforms of $\mu$-type works for subgroups of $A(\mathcal{X})$ is that the pushforwards along automorphisms are monoidal, i.e. $F(A \otimes B) \cong F(A) \otimes F(B)$ and $F(O_{\mathcal{X}}) = O_{\mathcal{X}}$ for $F = \phi_*$, and tensor products by line bundles are $\mathcal{X}$-linear, i.e. $G(A \otimes B) = G(A) \otimes B$ for $G = M_L$. If we consider varieties for simplicity, these are actually the only monoidal and linear autoequivalences. For the proof we use the following notion; compare [BO01] or [Huy06, Ch. 4].

Definition 1.3.11. Let $\mathcal{T}$ be a $\mathbb{C}$-linear triangulated category with Serre functor $S$. An object $E \in \mathcal{T}$ is called point-like if

1. $S(E) \cong E[\ell]$ for some $\ell \in \mathbb{Z}$,
2. $\text{Hom}(E, E[i]) = 0$ for $i < 0$, and
3. $E$ is simple, that is, $\text{Hom}(E, E) = \mathbb{C}$.

Proposition 1.3.12. Let $X$ and $Y$ be smooth projective varieties.

1. Every $X$-linear autoequivalence of $\mathcal{D}^b(X)$ is of the form $M_L[m]$ for some $L \in \text{Pic}X$ and $m \in \mathbb{Z}$.
2. Every monoidal autoequivalence $\mathcal{D}^b(X) \cong \mathcal{D}^b(Y)$ is of the form $\phi_*$ for some isomorphism $\phi: X \to Y$.

Proof. Let $G \in \text{Aut}(\mathcal{D}^b(X))$ be $X$-linear, $x \in X$, and $E := G(k(x))$. For $x \neq x' \in X$ we have $E \otimes k(x') \cong G(k(x)) \otimes k(x') \cong 0$. Thus, $\text{supp } E = \{x\}$; see [Huy06, Ex. 3.30]. Since $E$ is point-like, we have $E = k(x)[m]$ for some $m \in \mathbb{Z}$; see [Huy06, Lem. 4.5]. It follows that $G = M_L[m]$ for some line bundle $L \in \text{Pic}X$; see [Huy06, Cor. 5.23 & 6.14].

Let $F: \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ be a monoidal autoequivalence, $x \in X$, and $E := F(k(x))$. For every $A \in \mathcal{D}^b(X)$ we have $k(x) \otimes A \cong k(x) \otimes_{\mathbb{C}} V$ for some graded vector space $V \in \mathcal{D}^b(\mathbb{C})$ (namely $V = i_x^* E$ where $i_x$ is the embedding of the point). It follows for every $B \in \mathcal{D}^b(Y)$ that

$$E \otimes B \cong F(k(x) \otimes F^{-1}(B)) \cong F(k(x) \otimes_{\mathbb{C}} V) \cong E \otimes_{\mathbb{C}} V$$
for some $V \in \mathcal{D}^b(\mathbb{C})$. In particular, for every $B \in \mathcal{D}^b(Y)$ we have $\text{supp}(E \otimes B) = \text{supp}E$ or $\text{supp}(E \otimes B) = \emptyset$. Since $\text{supp}(E \otimes k(y)) \subset \{y\}$ for $y \in Y$, the support of $E$ consists of a single point. Again, since $E$ is point-wise, $E = k(y)[m]$ for some $y \in Y$ and $m \in \mathbb{Z}$. It follows again by [Huy06, Cor. 5.23 & 6.14] that $F = M_L \circ \varphi_*[m]$ for some $L \in \text{Pic}X$ and some isomorphism $\varphi : X \to Y$. Since $F$ is monoidal, $F(O_X) \cong O_Y$. Thus, $L = O_Y$ and $m = 0$.

**Remark 1.3.13.** Part (ii) of the above Proposition gives a quick proof that the derived category of a smooth projective variety together with its monoidal structure determines the variety (up to isomorphism); see [Bal05] for a constructive proof and a more general statement.

### 1.3.6 Lifts of adjunctions

**Definition 1.3.14.** Let $P \in \mathcal{D}^b(\mathcal{X} \times \mathcal{Y})$ and $Q \in \mathcal{D}^b_{\mu^{-1}}(\mathcal{Y} \times \mathcal{X})$. Then $Q$ is said to be a right adjoint kernel of $P$ (or, equivalently, $P$ is a left adjoint kernel of $Q$), in short $P \dashv Q$, if there are morphisms $\eta : O_{\Delta X}^\text{can} \to Q \ast P$ and $\varepsilon : P \ast Q \to O_{\Delta Y}^\text{can}$ with $(\varepsilon \ast \text{id}_P) \circ (\text{id}_P \ast \eta) = \text{id}_P$ and $(\text{id}_Q \ast \varepsilon) \circ (\eta \ast \text{id}_Q) = \text{id}_Q$. The morphisms $\eta$ and $\varepsilon$ are called the unit and counit of the adjunction.

Clearly, an adjunction of kernels induces an adjunction between the associated Fourier-Mukai transforms $\mathcal{F} \mathcal{M}^\mu_{P} \dashv \mathcal{F} \mathcal{M}^\mu_{Q}^{-1}$.

Let $Z$ be a third smooth projective stack, $H'' \subset A(Z)$, and $\mu' : H'' \cong H$ a c-isomorphism. Let $P \dashv Q$ be as in the definition. For $E \in \mathcal{D}^b(\mathcal{Z} \times \mathcal{X})$ and $F \in \mathcal{D}^b_{\mu^{-1}}(\mathcal{Z} \times \mathcal{Y})$ the unit and counit of the adjunction $P \dashv Q$ induce an isomorphism

$$\text{Hom}_{\mathcal{D}^b(\mathcal{Z} \times \mathcal{Y})}(P \ast E, F) \cong \text{Hom}_{\mathcal{D}^b_{\mu^{-1}}(\mathcal{Z} \times \mathcal{X})}(E, Q \ast F).$$

In particular, there is the formula

$$\text{Hom}_{\mathcal{D}^b(\mathcal{Z} \times \mathcal{Y})}(P, P) \cong \text{Hom}_{\mathcal{D}^b_{\mu^{-1}}(\mathcal{Z} \times \mathcal{X})}(O_{\Delta X}^\text{can}, Q \ast P).$$

Recall that every $P \in \mathcal{D}^b(\mathcal{X} \times \mathcal{Y})$ has a right and a left adjoint kernel given by $P^L := P^\vee \otimes \text{pr}_X^* \omega_\mathcal{Y}[\dim \mathcal{Y}]$ and $P^R := P^\vee \otimes \text{pr}_Y^* \omega_\mathcal{X}[\dim \mathcal{X}]$; see [CW10].

**Lemma 1.3.15.** Let $P \in \mathcal{D}^b(\mathcal{X} \times \mathcal{Y})$ be simple. Then for every $\mu$-linearisation $\nu$ of $P$, there exist $\mu^{-1}$-linearisations $\nu^L$ and $\nu^R$ of $P^L$ and $P^R$ such that $P^L := (P^L, \nu^L) \vdash P := (P, \nu) \vdash P^R := (P^R, \nu^R)$.

**Proof.** Let $\eta : O_{\Delta X} \to P^R \ast P$ and $\varepsilon : P \ast P^R \to O_{\Delta Y}$ be the unit and counit of the adjunction $P \dashv P^R$ and let $\nu$ be a linearisation of $P$. Note that $P$ being simple is equivalent to $\text{Hom}(O_{\Delta X}, P^R \ast P) = \mathbb{C}$ by (1.6). Furthermore, $P^R$ is always $\mu^{-1}$-linearisable; see the proof of Proposition 1.3.10. Let $\nu'$ be any $\mu^{-1}$-linearisation of $P^R$.

Consider the $H$-action on $\text{Hom}(O_{\Delta X}, P^R \ast P) = \mathbb{C}$ induced by the canonical linearisation of $O_{\Delta X}$ and the linearisation $\nu' \ast \nu$ of $P^R \ast P$. The $H$-action is given by some character $\varrho \in \hat{H}$. Thus, after replacing $\nu'$ by $\nu^R := \nu' \otimes \varrho$, the $H$-action on $\text{Hom}(O_{\Delta X}, P^R \ast P)$ becomes trivial which allows us to set $\tilde{\eta} := \eta : O_{\Delta X}^\text{can} \to P^R \ast P$. We also set $\tilde{\varepsilon} = \frac{1}{|H|} \sum_{h' \in H} h' \ast \varepsilon$ where $h' \ast \varepsilon$ is the image of $\varepsilon$ under the $H'$-action on $\text{Hom}(P \ast P^R, O_{\Delta Y})$ induced by the linearisation $\nu \ast \nu^R$.
and the canonical linearisation of $O_{\Delta Y}$. The equivariance of $\eta$ and $\text{id}_P$ yield
\[
(\tilde{\varepsilon} \ast \text{id}_P) \circ (\text{id}_P \ast \tilde{\eta}) = \frac{1}{|H|} \sum_{h \in H} (\mu(h) \cdot \varepsilon \ast \text{id}_P) \circ (\text{id}_P \ast \eta)
\]
\[
= \frac{1}{|H|} \sum_{h' \in H'} h' \cdot [(\varepsilon \ast \text{id}_P) \circ (\text{id}_P \ast \eta)]
\]
\[
= \frac{1}{|H|} \sum_{h' \in H'} h' \cdot \text{id}_P
\]
\[
= \text{id}_P.
\]
Similarly, we get $(\text{id}_{P_R} \ast \tilde{\varepsilon}) \circ (\tilde{\eta} \ast \text{id}_{P_R}) = \text{id}_{P_R}$. The proof of the existence of an adjunction $P_L \dashv P_R$ is analogous.

This allows us to extend Proposition 1.3.10 to the case of fully faithful functors.

**Proposition 1.3.16.** Let $\mathcal{P} = (P, \nu) \in D^b_h(X \times Y)$ for some isomorphism $\mu: H \to H'$, $F = \text{FM}_P$, and $\tilde{F} = \text{FM}_P^\mu$. Then:

\[
F \text{ is fully faithful} \quad \Rightarrow \quad \tilde{F} \text{ is fully faithful.}
\]

**Proof.** By the same arguments as in the proof of Proposition 1.3.10, we get $P_R \ast \mathcal{P} \cong O_{\Delta X}^\text{can} \otimes \varrho$ for some $\varrho \in \tilde{H}$. Since $O_{\Delta X} \in D^b(X \times X)$ is simple, we have $\text{Hom}_{D^b_h(X \times X)}(O_{\Delta X}^\text{can}, O_{\Delta X}^\text{can} \otimes \varrho) = 0$ for $\varrho \neq 1$. Hence, if $\varrho$ were non-trivial, we would get $\text{Hom}(\mathcal{P}, \mathcal{P}) = 0$ by (1.6) which is absurd. It follows that $P_R \ast \mathcal{P} \cong O_{\Delta X}^\text{can}$ so that $\tilde{F}$ is fully faithful. \qed

### 1.3.7 Lifts of spherical and $\mathbb{P}^n$-functors

**Definition 1.3.17.** An object $\mathcal{P} \in D^b_h(X \times Y)$ with left and right adjoints $P^L$ and $P^R$ is said to be a spherical kernel if the cone $\mathcal{C} = \text{cone}(\eta)$ of the unit is the kernel of an equivalence, called the cotwist kernel of $F = \text{FM}_P^\mu$, and the composition

\[
\varphi: P_R \xrightarrow{P_R \ast \eta_L} P_R \ast \mathcal{P} \ast P_L \xrightarrow{\beta \ast P_L} \mathcal{C} \ast P_L,
\]

(\text{where } \eta_L \text{ is the unit of the adjunction } P^L \dashv \mathcal{P}, \ \beta \text{ is from the triangle } O_{\Delta X}^\text{can} \xrightarrow{\eta} P^R \ast \mathcal{P} \xrightarrow{\beta} \mathcal{C} \text{ and we abuse notation by writing } P^R \text{ for } \text{id}_{P_R} \text{ and similarly for } P^L) \text{ is an isomorphism.}

If $\mathcal{P}$ is spherical, the associated twist $T = \text{cone}(\varepsilon) \in D^b_h(Y \times Y)$ given by the counit of the adjunction $\mathcal{P} \dashv P^R$ is the kernel of an auto-equivalence; see [Rou06], [Add16], [AL12].

Furthermore, following [Add16] we have

**Definition 1.3.18.** An object $\mathcal{P} \in D^b_h(X \times Y)$ is a $\mathbb{P}^n$-kernel if the following three conditions hold:

1. There is an isomorphism $\alpha: P^R \ast \mathcal{P} \cong O_{\Delta X}^\text{can} \oplus D \oplus D^* \oplus \ldots \oplus D^{*n}$ for some autoequivalence kernel $D \in D^b_{id_H}(X \times X)$ which is then called the $\mathbb{P}$-cotwist kernel. The components of the isomorphism $\alpha$ are denoted by $\alpha_i: P^R \ast \mathcal{P} \to D^{*i}$. \hfill 48
2. The composition

$$\psi: \mathcal{P}^R \xrightarrow{\mathcal{P}^R \ast \eta_{\mathcal{L}}} \mathcal{P} \ast \mathcal{P} \ast \mathcal{P} \xrightarrow{\alpha_n \ast \mathcal{P}^L} \mathcal{D}^n \ast \mathcal{P}^L$$

is an isomorphism.

3. The compositions

$$c_{ij}: \mathcal{D} \ast \mathcal{D}^i \rightarrow \mathcal{P} \ast \mathcal{P} \ast \mathcal{P} \ast \mathcal{P} \ast \mathcal{P} \ast \mathcal{P} \ast \mathcal{P} \ast \mathcal{P} \xrightarrow{\alpha_j} \mathcal{D}^j$$

are isomorphisms for $i = j$ and zero for $i < j$ (and arbitrary for $i > j$).

For $\mathbb{P}^n$-kernels there also is an associated twist which is an autoequivalence of $\mathcal{D}^b_{H'}(\mathcal{Y})$; see [Add16, Sect. 3.3]. Clearly, the above definitions include the case that $H = H' = 1$. The Fourier–Mukai transforms associated to spherical and $\mathbb{P}^n$-kernels are called spherical and $\mathbb{P}^n$-functors, respectively.

In Proposition 1.3.21 below we will assume that our spherical and $\mathbb{P}^n$-functors satisfy the conditions

$$\text{Hom}^{\leq 0}(\mathcal{O}^{\text{can}}_{\Delta X}, \mathcal{C}) = 0 \quad \text{for } \mathcal{P} \text{ a spherical kernel},$$

$$\text{Hom}^{\leq 0}(\mathcal{O}^{\text{can}}_{\Delta X}, \mathcal{D}^i) = 0 \quad \forall 1 \leq i \leq n \quad \text{for } \mathcal{P} \text{ a } \mathbb{P}^n\text{-kernel.} \quad (1.10)$$

Note that every spherical kernel satisfying (1.9) as well as every $\mathbb{P}^n$-kernel satisfying (1.10) is simple by (1.6). Furthermore, under condition (1.10) it is automatic that $c_{ij} = 0$ for $i < j$.

**Remark 1.3.19.** All spherical and $\mathbb{P}^n$-kernels the authors are aware of satisfy the conditions (1.9) and (1.10) as well as the slightly stronger conditions

$$\text{Hom}^{\leq 0}(\mathcal{O}^{\text{can}}_{\Delta X}, \mathcal{C}) = 0 \quad \text{for } \mathcal{P} \text{ a spherical kernel},$$

$$\text{Hom}^{\leq 0}(\mathcal{O}^{\text{can}}_{\Delta X}, \mathcal{D}^i) = 0 \quad \forall 1 \leq i \leq n \quad \text{for } \mathcal{P} \text{ a } \mathbb{P}^n\text{-kernel.} \quad (1.12)$$

where $\text{Hom}^{\leq 0}(E, F) = \oplus_{i \leq 0}(E, F[i])$. By (1.5), for every spherical kernel satisfying (1.11) as well as for every $\mathbb{P}^n$-kernel satisfying (1.12), the associated Fourier–Mukai transform satisfies (1.3). We will use this fact in Remarks 1.3.23 and 1.4.4.

**Remark 1.3.20.** The above axiom (ii) appears to be slightly stronger than in [Add16] where it is only required that there is any isomorphism $\mathcal{P}^R \cong \mathcal{D}^{\ast n} \ast \mathcal{P}^L$ (and in the spherical case it is only required that there is any isomorphism $\mathcal{P}^R \cong \mathcal{C} \ast \mathcal{P}^L$). However, if $\mathcal{P}$ is simple and satisfies the condition

$$\text{Hom}(\mathcal{O}^{\text{can}}_{\Delta X}, \mathcal{C}) = 0 = \text{Hom}(\mathcal{C}^{\ast -1}, \mathcal{C}) \quad \text{for } \mathcal{P} \text{ a spherical kernel} \quad (1.13)$$

$$\text{Hom}(\mathcal{D}^i, \mathcal{D}^{\ast n}) = 0 \quad \forall -n \leq i \leq n - 1 \quad \text{for } \mathcal{P} \text{ a } \mathbb{P}^n\text{-kernel} \quad (1.14)$$

(where $\mathcal{C}^{\ast -1}$ denotes the kernel of the autoequivalence which is inverse to $\mathcal{C}$) as most known spherical and $\mathbb{P}^n$-functors do, both definitions are equivalent (note that there are spherical functors satisfying (1.9) but not (1.13); see [3]). Indeed, let $\mathcal{P} \in \mathcal{D}^b(\mathcal{X} \times \mathcal{Y})$ satisfy axiom (i) of a $\mathbb{P}^n$-functor and let there be an isomorphism $\vartheta: \mathcal{P}^R \cong \mathcal{D}^{\ast n} \ast \mathcal{P}^L$. Then, for $\psi: \mathcal{P}^R \rightarrow \mathcal{D}^{\ast n} \ast \mathcal{P}^L$
to be an isomorphism it is sufficient that \( \psi \) is non-zero because \( P^R \) is simple. For \( i < j \) set 
\[ D^{[i,j]} := D^{*i} \oplus D^{*i+1} \oplus \ldots \oplus D^{*j} \]
and consider the triangle 
\[ D^{*[0,n-1]} \star P^L \to P^R \star P \star P^L \xrightarrow{\alpha \star \eta \star \rho} D^{*n} \star P^L. \]
Under the assumption \( \psi = (\alpha \star P^L) \circ (P^R \star \eta_L) = 0 \) it would follow that \( P^R \star \eta_L \) factors over 
\[ D^{*[0,n-1]} \star P^L. \] Thus, the identity \( \text{id}_{P^R \star P} = (P^R \star P \star \varepsilon_L) \circ (P^R \star \eta_L \star P) \) would factor over 
\[ D^{*[0,n-1]} \star P^L \star P \cong D^{*[0,n-1]} \star D^{*n} \star P^R \cong D^{*[0,n-1]} \star D^{*n} \star D^{*[0,n]} \]
which is impossible by condition \((1.14)\).

**Proposition 1.3.21.** Let \( P \in D^b(\mathcal{X} \times \mathcal{Y}) \) be a spherical kernel satisfying \((1.9)\) (a \( \mathbb{P}^n \)-kernel satisfying \((1.10)\)) which allows a \( \mu \)-linearisation \( \nu \). Then \( P := (P, \nu) \) is again a spherical \( (\mathbb{P}^n \cdot) \) kernel and also satisfies the respective condition.

**Proof.** We give the proof only in the case that \( P \) is a \( \mathbb{P}^n \)-kernel since the proof in the spherical case is similar. Let \( P \) be a \( \mathbb{P}^n \)-kernel with \( \mathbb{P} \)-cotwist kernel \( D \) and an isomorphism \( \alpha: P^R \star P \cong \mathcal{O}_{\Delta X} \oplus D \oplus \ldots \oplus D^{*n} \) and let \( \nu \) be a \( \mu \)-linearisation of \( P \). For \( i < j \) and \( h \in H \) there do not exist non-zero morphisms \( D^{*i} \to (h \times (h^{-1})^* D^{*j}). \) Indeed, applying \( (h^{-1} \times (h^{-1}))^* \circ D^{*i} \) to such a morphism would yield a non-zero morphism \( \mathcal{O}_{\Delta X} \cong (h^{-1} \times (h^{-1}))^* \mathcal{O}_{\Delta X} \to D^{*j-i} \)
contradicting \((1.10)\). It follows that the linearisation \( \nu^R \star \nu \) of \( P^R \star P \) (see Lemma 1.3.15) induces via the isomorphism \( \alpha \) linearisations of the \( D^{*i} \). We set \( \mathcal{P} := (P, \nu), P^R := (P^R, \nu^R), \) and denote for \( i = 1, \ldots, n \) the \( D^{*i} \) equipped with the induced linearisations by \( D_i \), so that \( \alpha \) gives an isomorphism \( P^R \star \mathcal{P} \cong \mathcal{O}_{\Delta X}^{\text{can}} \oplus D_1 \oplus \ldots \oplus D_n \) in \( D^b_{\text{id}_H}(\mathcal{X} \times \mathcal{X}) \). The \( D_i \) are equivalence kernels by Proposition 1.3.10(ii). By the definition of the linearisations of the \( D^{*i} \) the projections \( \alpha_i: P^R \star \mathcal{P} \to D^{*i} \) are equivariant. Since the \( c_{ii} \) are given by a composition of the \( \alpha_i \), their inverses, and counits of adjunctions, the \( c_{ii} \) are equivariant, too. Hence, they can be seen as isomorphisms \( c_{ii}: D_1 \star D_{i-1} \cong D_i \) in \( D^b_{\text{id}_H}(\mathcal{X} \times \mathcal{X}) \). This shows by induction that \( D_i \cong D^{*i} \), where \( D := D_1 \), so that \( \mathcal{P} \) satisfies condition (i) of a \( \mathbb{P}^n \)-kernel. It also shows that condition (iii) of a \( \mathbb{P}^n \)-kernel holds. The morphism \( \psi \) associated to \( \mathcal{P} \) is just the same as the morphism \( \psi \) associated to \( P \), since also the isomorphism \( \alpha \) is the same in the equivariant and the non-equivariant case. In particular, \( \psi \) is an isomorphism and \( \mathcal{P} \) fulfills condition (ii) of a \( \mathbb{P} \)-functor. Finally, \( \text{Hom}(\mathcal{O}_{\Delta X}^{\text{can}}, D^{*i}) \subset \text{Hom}(\mathcal{O}_{\Delta X}, D^{*i}) \) so that condition \((1.10)\) is still satisfied.

Summarising the results of this section, we have

**Theorem 1.3.22.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be smooth projective stacks, let \( H \subset A(\mathcal{X}) \) and \( H' \subset A(\mathcal{Y}) \) be \( c \)-invariant finite subgroups acting categorically, and let \( P \in D^b(\mathcal{X} \times \mathcal{Y}) \). If \( P \) admits a \( \mu \)-linearisation \( \nu \) for some \( \mu \)-isomorphism \( \mu: H \to H' \), then \( F = FM_{(P, \nu)}: D^b(\mathcal{X}) \to D^b(\mathcal{Y}) \) lifts to \( \tilde{F} = FM_{(P, \nu)}^H : D^b_H(\mathcal{X}) \to D^b_H(\mathcal{Y}). \) If \( F \) is fully faithful (an equivalence, a spherical kernel satisfying \((1.9)\) or a \( \mathbb{P}^n \)-kernel satisfying \((1.10)\)), then the same holds for \( \tilde{F} \).

**Proof.** This is just a combination of Lemma 1.3.9 together with Propositions 1.3.10, 1.3.16, and 1.3.21.

**Remark 1.3.23.** Let \( P \in D^b(\mathcal{X} \times \mathcal{Y}) \) be a kernel of an equivalence, a fully faithful functor, a spherical kernel satisfying \((1.11)\), or a \( \mathbb{P}^n \)-kernel satisfying \((1.12)\). Then, as pointed out
in Remark 1.3.19, $P$ is simple and the functor $F = \text{FM}_P$ satisfies condition (1.3). Thus, if $H$ is cyclic, the functor $F = \text{FM}_P$ lifts as soon as it is $\mu$-equivariant; see Lemma 1.3.6. Furthermore, note that in this case $P$ has exactly $|H|$ different $\mu$-linearisations. The induced lifts $\hat{F}: D^b_H(X) \to D^b_H(Y)$ differ by $M_\varrho$ for $\varrho \in H$.

1.4 Geometric Interpretation

By a Galois cover with group of deck transformations $H$ of smooth projective stacks we mean a quotient morphism $\pi: \tilde{X} \rightarrow X := [\tilde{X}/H]$ for a finite subgroup $H \subset \text{Aut}(\tilde{X})$. For details on quotients of stacks by group actions, we refer to [Rom05].

**Definition 1.4.1.** Let $\pi: \tilde{X} \rightarrow X$ and $\pi': \tilde{Y} \rightarrow Y$ be Galois covers with groups of deck transformation $H$ and $H'$, respectively, and $\tilde{F}: D^b(\tilde{X}) \to D^b(\tilde{Y})$ and $F: D^b(X) \to D^b(Y)$ be exact functors. Then $F$ is called a descent of $\tilde{F}$ (and, equivalently, $\tilde{F}$ is called a lift of $F$) if the following two conditions hold:

$$F \circ \pi_* \simeq \pi'_* \circ \tilde{F}: D^b(X) \to D^b(Y), \quad \pi'^* \circ F \simeq \tilde{F} \circ \pi^*: D^b(X) \to D^b(Y).$$

By [Rom05, Thm. 4.1], the cover $\pi: \tilde{X} \rightarrow X = [\tilde{X}/H]$ is an $H$-torsor. Thus, $\text{Coh}(X) \cong \text{Coh}_H(\tilde{X})$; see [Vis05, Thm. 4.46]. Under this isomorphism, $\pi^*$ corresponds to $\text{Res}$ and $\pi_*$ corresponds to $\text{Inf}$. Of course, similar considerations apply to $Y$. Thus, for a functor $F: D^b(X) \to D^b(Y)$ a descent $D^b(X) \to D^b(Y)$ in the sense of Definition 1.4.1 is the same as an equivariant lift $D^b_H(X) \to D^b_H(Y)$ in the sense of Definition 1.3.7. Furthermore, the objects $O^\text{can}_{\tilde{X}} \otimes \varrho$ for $\varrho \in H$ correspond to the line bundles on $[\tilde{X}/H]$ which are in the kernel of $\pi^*: \text{Pic}([\tilde{X}/H]) \rightarrow \text{Pic}(\tilde{X})$; compare [Mum70, Sect. 7]. Hence, we can identify $H$ with the subgroup $\text{ker}(\pi^*) \subset \text{Pic}(\tilde{X}/H)$. Combining the above with Theorem 1.3.22 immediately gives

**Proposition 1.4.2.** Let $\tilde{X} \rightarrow X$ and $\tilde{Y} \rightarrow Y$ be Galois covers with groups of deck transformations $H$ and $H'$, respectively. Let $\Phi = \text{FM}_P: D^b(\tilde{X}) \to D^b(\tilde{Y})$ be a Fourier-Mukai transform. If $P \in D^b(\tilde{X} \times \tilde{Y})$ is $\mu$-linearisable for some isomorphism $\mu: H \cong H'$, then there is a $\mu$-equivariant descent $\Psi: D^b(X) \to D^b(Y)$. In addition, if $\Phi$ is an equivalence (fully faithful, spherical satisfying (1.9), $\mathbb{P}^n$ satisfying (1.10)), the same holds for $\Psi$. \hfill $\square$

If $\pi: \tilde{X} \rightarrow X$ is a Galois cover with an abelian group of deck transformations $H$, we also have $\text{Coh}(X) \cong \text{Coh}_H(X)$ by Proposition 1.2.2. Note that in this situation $\pi_* = \text{Res}$ and $\pi^*$ is $\text{Inf}$. The results of the previous subsection again apply and give

**Proposition 1.4.3.** Let $\pi: \tilde{X} \rightarrow X$ and $\pi': \tilde{Y} \rightarrow Y$ be Galois covers with abelian groups of deck transformations $H$ and $H'$, respectively. Let $\Phi = \text{FM}_P: D^b(X) \to D^b(Y)$ be a Fourier-Mukai transform. If $P \in D^b(X \times Y)$ is $\mu$-linearisable for some isomorphism $\mu: H \cong H'$, there is a $\mu$-equivariant lift $\Psi: D^b(X) \to D^b(Y)$. In addition, if $\Phi$ is an equivalence (fully faithful, spherical satisfying (1.9), $\mathbb{P}^n$ satisfying (1.10)), the same holds for $\Psi$. \hfill $\square$

**Remark 1.4.4.** Let us assume that $H$ and $H'$ are cyclic and, in the cases of spherical and $\mathbb{P}^n$-functors, that the stronger conditions (1.11) and (1.12) hold. Then one can replace the condition that $P$ is $\mu$-linearisable in Propositions 1.4.2 and 1.4.3 by the weaker condition that $\Phi$ is $\mu$-equivariant; see Remark 1.3.23.
Conversely, let $\mathcal{X}$ be a smooth projective stack and $H \subset \text{Pic} \mathcal{X}$ a finite subgroup. Let $\mathcal{X}_H := \text{Spec}(\mathcal{A}_H)$ where $\mathcal{A}_H = \bigoplus_{L \in H} \mathcal{L}$ is an $\mathcal{O}_\mathcal{X}$-algebra with multiplication given by tensor product. The group of characters $H$ acts on $\mathcal{A}_H$. The action of $\varrho \in H$ is multiplication by $\varrho(L)$ on the summands $L$ of $\mathcal{A}$. The induced action on $\mathcal{X}_H$ satisfies $[\mathcal{X}_H/H] \cong \mathcal{X}$, so that $\mathcal{X}_H$ is a Galois cover with $H$ as the group of deck transformations.

In particular, if the canonical bundle of $\mathcal{X}$ is torsion of order $n$, there is the canonical cover $\tilde{\mathcal{X}} := \mathcal{X}_{\omega} := \mathcal{X}(\omega_{\mathcal{X}})$. It is a cyclic Galois cover of order $n$ of $\mathcal{X}$ with trivial canonical bundle.

Note that if $\mathcal{Y}$ is another smooth projective stack together with an equivalence $D^b(\mathcal{X}) \cong D^b(\mathcal{Y})$, then the canonical bundle of $\mathcal{Y}$ is also torsion of order $n$. This follows from the fact that equivalences commute with Serre functors; compare [Huy06, Prop. 1.30].

**Corollary 1.4.5.** If $\mathcal{X}$ and $\mathcal{Y}$ are smooth projective stacks with torsion canonical bundles, then every equivalence $\Phi: D^b(\mathcal{X}) \to D^b(\mathcal{Y})$ lifts to an equivalence $\Psi: D^b(\tilde{\mathcal{X}}) \to D^b(\tilde{\mathcal{Y}})$ between the canonical covers.

**Proof.** Since the equivalence $\Phi$ commutes with the Serre functors and shifts, we have $M_{\omega_{\mathcal{X}}} \circ \Phi \cong \Phi \circ M_{\omega_{\mathcal{Y}}}$ for every $i \in \mathbb{Z}$. This means that $\Phi$ is $\mu$-equivariant where $\mu: (\omega_{\mathcal{X}}) \cong (\omega_{\mathcal{Y}})$ is the isomorphism sending $\omega_{\mathcal{X}}$ to $\omega_{\mathcal{Y}}$. The assertion follows by Proposition 1.4.3 and Remark 1.4.4.

**Remark 1.4.6.** Note that if $\mathcal{X} = X$ is a variety, the quotient $[X/H]$ is a variety (namely the quotient variety) if and only if $H$ acts freely on $X$. Thus, for varieties our notion of Galois covers agrees with the usual one. In particular, the criteria of [BM98] and [LP15] for descent and lift along cyclic covers are obtained as special cases of Propositions 1.4.2 and 1.4.3, respectively.

### 1.5 Applications

#### 1.5.1 Hochschild homology

Let $G$ be a finite group acting faithfully on a smooth projective variety $X$ of dimension $d$, let $\mathcal{X} = [X/G]$ be the associated global quotient stack and write $\omega_{\Delta \mathcal{X}}$ for $\Delta_* \omega_{\mathcal{X}}$. Let $\Sigma^X := \omega_{\Delta \mathcal{X}}[d] \in D^b(\mathcal{X} \times \mathcal{X})$ be the kernel of the Serre functor.

The Hochschild homology $\mathcal{H}H_*(\mathcal{X}) = \text{Hom}^*(\Sigma^{-1}, \mathcal{O}_{\Delta \mathcal{X}})$ decomposes as

$$\mathcal{H}H_*(\mathcal{X}) \cong \bigoplus_{g \in \text{conj}(G)} \mathcal{H}H_*(X^g)^{C(g)}; \quad (1.15)$$

see, for example, [Pop13, Sect. 3]. Here $\text{conj}(G)$ denotes a set of representatives of the conjugacy classes in $G$, $C(g) := C_G(g)$ is the centraliser of $g$ in $G$, and $X^g \subset X$ is the fixed point locus. The decomposition can be obtained by the following computation, in which we use the notation $G_{\Delta}$ for the diagonal action of $G$ on $X \times X$ and the fact that the centraliser of an element $g \in G$ is its stabiliser with respect to the conjugation action of $G$ on itself. Note that...
The components of $\nu$ by sending $\nu$. Thus, the components of $U$ is the projection to the summands indexed by $U$. The first morphism is the inclusion of the summands indexed by $U$. The pull-back $\nu$ satisfies condition (1.3) (so that the FM kernel is unique), it also makes sense to speak of the pull-back $\nu^*: \mathcal{H}_s(Y) \to \mathcal{H}_s(X)$ on Hochschild homology along the kernel $\nu$ is defined by sending $\nu \in \mathcal{H}_s(Y) = \text{Hom}^*(\Sigma^{-1}, \mathcal{O}_X)$ to the composition

$$
\Sigma^{-1} \xrightarrow{\nu^* \eta} \Sigma^{-1} \star \nu \star \nu \star \Sigma^{-1} \xrightarrow{\nu^* \nu \nu \nu} \nu \star \nu \star \nu \star \nu \xrightarrow{\nu^* \nu \nu \nu} \mathcal{O}_X
$$

(1.17)

(here the shift by the degree of $\nu$ is omitted in the notation); see [CW10, Sect. 4.3]. If $F = \text{FM}_\nu$ satisfies condition (1.3) (so that the FM kernel is unique), it also makes sense to speak of the pull-back $F^* := \nu^*: \mathcal{H}_s(Y) \to \mathcal{H}_s(X)$ along $F$ instead.

In the following let $U \leq G$ be a subgroup.

**Lemma 1.5.1.** Under the isomorphism (1.15), the only non-zero components of the pull-back $\text{Inf}^*: \mathcal{H}_s([X/G]) \to \mathcal{H}_s([X/U])$ along the inflation functor $\text{Inf}_U^*: \mathcal{D}_U^b(X) \to \mathcal{D}_G^b(X)$ are those of the form

$$\mathcal{H}_s(X^u)^{C_{G^u}}(u) \to \mathcal{H}_s(X^u)^{C_{U^u}}(u) \quad \text{for } u \in U.$$ 

These components are given by the embedding of invariants.

**Proof.** The FM kernel of $\text{Inf} = \text{Inf}^G_U$ is given by

$$\nu = \text{Inf}_U^G \mathcal{O}_X = \oplus_{g \in G} \mathcal{O}_{\Gamma_g} \in \mathcal{D}_U^b(X \times X).$$

The adjoint kernels are the same as $\nu$ just considered as an object of $\mathcal{D}_G^b(X \times X)$ instead, that is, $\nu^L = \nu^R = \oplus_{g \in G} \mathcal{O}_{\Gamma_g} \in \mathcal{D}_G^b(X \times X)$. Hence, in our case the composition (1.17) is given by

$$\bigoplus_{u \in U} \omega_{\Gamma_u}^{-1}[-d] \to \bigoplus_{g \in G} \omega_{\Gamma_g}^{-1}[-d] \xrightarrow{\nu^* \nu \nu \nu} \bigoplus_{g \in G} \mathcal{O}_{\Gamma_g} \to \bigoplus_{u \in U} \mathcal{O}_{\Gamma_u}.$$

The first morphism is the inclusion of the summands indexed by $U$ and the last morphism is the projection to the summands indexed by $U$. This follows from the fact that these are, up to multiplication by a scalar, the only non-vanishing $U \times U$-equivariant morphisms. Thus, the components of $\text{Inf}^*\nu$ coincide with the components of $\nu^L \nu^R$ on the summands indexed by $U$. Now, the assertion of the lemma can be confirmed by following an element $\nu \in \text{Hom}_*(\omega_{\Delta}^{-1}, \mathcal{O}_{\Gamma_g})^C(g) \cong \mathcal{H}_s(X^g)^{C(g)} \subset \mathcal{H}_s([X/G])$ through the isomorphisms of (1.16). \qed
Corollary 1.5.2. The pull-back $\text{Inf}^*: \text{HH}_s([X/G]) \to \text{HH}_s([X/U])$ is injective if and only if all elements of $G \setminus U$ act freely on $X$.

Proof. If all $g \in G \setminus U$ act freely, the corresponding fixed point loci $X^g$ are empty and thus all the components $\text{HH}_s(X^g)^{C(g)}$ in the kernel of $\text{Inf}^*$ vanish. Conversely, if $g \in G \setminus U$ does not act freely, by the Hochschild-Kostant-Rosenberg isomorphism

$$\text{HH}_s(X^g)^{C(g)} \cong H^*(X^g, \mathbb{C})^{C(g)} \cong H^*(X^g / C(g), \mathbb{C}) \neq 0$$

(note that the HKR isomorphism is not graded in the usual sense but this does not cause problems since we only need the non-vanishing).

Remark 1.5.3. The case $U = 1$ says that $\text{Inf}: \text{HH}_s([X/G]) \to \text{HH}_s(X)$ is injective if and only if the quotient stack $[X/G]$ agrees with the quotient variety $X/G$. Note that in this case $\text{Inf}$ corresponds to the pushforward $\pi_*: \text{D}^b(X) \to \text{D}^b(X/G)$ along the quotient morphism $\pi: X \to X/G$ under the equivalence $\text{D}^b_G(X) \cong \text{D}^b(X/G)$.

1.5.2 Galois coverings induced by characters

Let $G$ be a finite group acting on a smooth projective variety $X$. For a character $\varrho \in \hat{G}$, let $U := U_\varrho := \ker(\varrho: G \to \mathbb{C}^*)$. Then $U < G$ is a normal subgroup of index $n := \text{ord} \varrho$ and the canonical morphism $[X/U] \to [X/G]$ is the Galois covering induced by the line bundle $\mathcal{O}_X \otimes \varrho \in \text{Pic}([X/G])$.

Proposition 1.5.4. Let $Y$ be a smooth projective variety and $\Phi: \text{D}^b([X/G]) \to \text{D}^b(Y)$ an equivalence. Let $\varrho \in \hat{G}$ be a character, $U = \ker(\varrho: G \to \mathbb{C}^*)$, and assume that there is an element of $G \setminus U$ that does not act freely on $X$. Then the induced autoequivalence $\Phi \circ M_{\varrho} \circ \Phi^{-1}$ of $\text{D}^b(Y)$ is not given by tensor product with a line bundle on $Y$.

Proof. Assume that $\Phi \circ M_{\varrho} \circ \Phi^{-1} \cong M_L$ for some line bundle $L \in \text{Pic}(Y)$ and let $\pi: Y_L \to Y$ be the associated Galois cover. Clearly, $\text{ord}(L) = \text{ord}(\varrho)$ so that there is the isomorphism $\mu: \langle \varrho \rangle \cong \langle L \rangle$. Our assumption says that the equivalence $\Phi$ is $\mu$-equivariant. Thus, by Proposition 1.4.3 it lifts to an equivalence $\text{D}^b([X/U]) \cong \text{D}^b(Y_L)$ which means that there is a commutative diagram

$$\begin{array}{ccc}
\text{D}^b([X/U]) & \cong & \text{D}^b(Y_L) \\
\text{Inf} & & \pi_* \\
\text{D}^b([X/G]) & \cong & \text{D}^b(Y).
\end{array}$$ (1.18)

Corollary 1.5.2 says that $\text{Inf}^*: \text{HH}^*([X/G]) \to \text{HH}^*([X/U])$ is not injective. On the other hand, $(\pi_*)^*: \text{HH}^* (Y) \to \text{HH}^* (Y_L)$ is injective by Remark 1.5.3. Considering the commutative diagram (1.18), this contradicts the functoriality of the pull-back in Hochschild homology; see [CW10, Thm. 6].

Remark 1.5.5. When the action of $G$ on $X$ satisfies the assumptions of the Bridgeland-King-Reid theorem [BKR01], the above yields non-standard autoequivalences of the crepant resolution $Y = \text{Hilb}^G(X)$ of the quotient variety $X/G$. Indeed, let $\varrho \in \hat{G}$ be a character and let $F_{\varrho} := \Phi \circ M_{\varrho} \circ \Phi^{-1}$ where $\Phi: \text{D}^b([X/G]) \cong \text{D}^b(Y)$ is the BKR equivalence. Assume that
is standard, that is \( F_\varphi = M_L \circ \varphi_* m \) for some \( L \in \text{Pic}(Y) \), \( \varphi \in \text{Aut}(Y) \), and \( m \in \mathbb{Z} \). There is an open subset \( U \subset Y \) such that skyscraper sheaves of points on \( U \) correspond under \( \Phi \) to skyscraper sheaves of free orbits. Since skyscraper sheaves of free orbits lie in the image of the inflation functor, they are invariant under \( M_\varphi \). It follows that \( F_\varphi(\mathcal{C}(u)) = \mathcal{C}(u) \) for \( u \in U \) which shows that \( \varphi = \text{id} \) and \( m = 0 \). Thus, we are left with \( F_\varphi = M_L \) which is ruled out (in the case that \( G \setminus U_\varphi \) does not act trivially) by the above proposition.

The above also works in the more general setting of \( G \)-constellations, see, for instance, \([CI04]\) for this notion.

### 1.5.3 Stacks with characters as canonical bundles

**Proposition 1.5.6.** Let \( X \) be a smooth projective variety with trivial canonical bundle and \( G \subset \text{Aut}(X) \) a finite subgroup such that the canonical bundle is not trivial as a \( G \)-bundle, that is, \( \omega_{[X/G]} \cong \mathcal{O}_X \otimes \varrho \) for some non-trivial character \( \varrho \). Then there is no smooth projective variety \( Y \) such that \( \mathcal{D}^b([X/G]) \cong \mathcal{D}^b(Y) \).

**Proof.** Assume that there exists a smooth projective variety \( Y \) and an equivalence \( \Phi: \mathcal{D}^b([X/G]) \cong \mathcal{D}^b(Y) \). Since any equivalence commutes with shifts and Serre functors, we have \( \Phi \circ M_{\varrho} \circ \Phi^{-1} \cong M_{\varrho} \) which contradicts Proposition 1.5.4. \( \square \)

By a completely different argument we get a similar statement in the case that \( \omega_X \) is (anti-) ample:

**Proposition 1.5.7.** Let \( X \) be a smooth projective variety such that \( \omega_X \) or its dual \( \omega_X^{-1} \) is ample. Let \( G \subset \text{Aut}(X) \) a finite subgroup such that there exists a point \( x \in X \) with the property that \( G_x \) is a non-trivial cyclic group and its action on the fibre \( \omega_X(x) \) is given by a generator \( \varrho \) of \( G_x \). Then there is no smooth projective variety \( Y \) such that \( \mathcal{D}^b([X/G]) \cong \mathcal{D}^b(Y) \).

**Proof.** All skyscraper sheaves of points on a smooth projective variety \( Y \) are point-like in the sense of Definition 1.3.11. Hence, the point-like objects form a spanning class of \( \mathcal{D}^b(Y) \). On the other hand, by Lemma 1.5.8 below, the point-like objects in \( \mathcal{D}^b([X/G]) \cong \mathcal{D}^b_G(X) \) do not form a spanning class. The assertion follows, since exact equivalences preserve point-like objects as well as spanning classes. \( \square \)

**Lemma 1.5.8.** Let \( X \) and \( G \subset \text{Aut}(X) \) be as in Proposition 1.5.7. Then for every point-like object \( E \in \mathcal{D}^b_G(X) \), we have \( x \notin \text{supp} E \).

**Proof.** Condition (i) of a point-like object together with the (anti-)ampleness of the canonical bundle implies that \( \text{supp} E \) is zero-dimensional; compare \([Huy06\text{, proof of Prop. 4.6}]\). Since \( E \) is simple, it is supported on a single orbit \( G \cdot y \). Hence, it is of the form \( E \cong \text{Inf}_{G_y} B \) for some \( A \in \mathcal{D}^b_{G_y}(X) \) supported in \( \{y\} \). By the adjunction \( \text{Inf} \dashv \text{Res} \), the object \( A \) has to be point-like too.

Thus, it is sufficient to show that there are no point-like objects \( A \in \mathcal{D}^b_{G_x}(X) \) with \( \text{supp} A = \{x\} \). If \( \text{supp} A = \{x\} \), we have \( S(A) = A \otimes g[\dim X] \). Hence, for \( A \) to be point-like it needs to be \( \hat{G}_x \)-linearisable; see Lemma 1.2.3. Thus, \( A \cong \text{Inf}_{G_x} B \) for some \( B \in \mathcal{D}^b(X) \); see Proposition 1.2.2. Again, for \( A \in \mathcal{D}^b_{G_x}(X) \) to be point-like, \( B \in \mathcal{D}^b(X) \) needs to be point-like too. By the (anti-)ampleness of \( \omega_X \), we have \( B = \mathcal{C}(x)[m] \) for some \( m \in \mathbb{Z} \); see \([Huy06\text{, Prop. 4.6}]\). Because of \( \text{Inf} \dashv \text{Res} \) we get

\[
\text{Hom}(A, A) \cong \bigoplus_{g \in G_x} \text{Hom}(B, g^* B) \cong \bigoplus_{g \in G_x} \text{Hom}(\mathcal{C}(x), \mathcal{C}(x))
\]
so that $A$ cannot be simple.

In some vague sense Propositions 1.5.6 and 1.5.7 can be seen as partial converses of the Bridgeland-King-Reid theorem [BKR01] which gives conditions which are sufficient for $[X/G]$ to be derived equivalent to a special smooth variety, namely the Nakamura G-Hilbert scheme. One of the two conditions is that $\omega_X$ is locally trivial as a $G$-bundle. The above says that, under the additional assumption that $\omega_X$ is trivial or (anti-)ample as a non-equivariant bundle, this condition is also necessary for $[X/G]$ to be derived equivalent to any smooth projective variety. This leads to the question whether the same is true without any requirements on canonical bundle, that is

**Question 1.5.9.** Let $X$ be a smooth projective variety and $G \subset \text{Aut}(X)$ a finite subgroup such that $\omega_X$ is not locally trivial as a $G$-bundle. Can there be a smooth projective variety $Y$ such that $\mathcal{D}^b([X/G]) \cong \mathcal{D}^b(Y)$?

### 1.5.4 Symmetric quotients and generalised Kummer stacks

We introduce our two main example series. Let $X$ be any smooth projective variety and $n \geq 2$. The symmetric group $S_n$ acts on $X^n$ by permutation of the factors. We call the corresponding global quotient stack $S^nX := [X^n/S_n]$ the symmetric quotient stack. The most relevant case is the one when $X$ is a surface, where $S^nX$ is derived equivalent to the Hilbert scheme $X^{[n]}$ of $n$ points on $X$ by [BKR01] and [Hai01]; see Subsection 1.5.6 for details.

In the case that $X = A$ is an abelian variety, $A^n$ contains the $S_n$-invariant subvariety

$$N_{n-1}A := \{(a_1, \ldots, a_n) \mid a_1 + \ldots + a_n = 0\} \cong A^{n-1}.$$ 

We call $K_{n-1}A := [N_{n-1}A/S_n]$ the generalised Kummer stack. The reason for the name is that in the case that $A$ is an abelian surface, $K_{n-1}A$ is derived equivalent to the generalised Kummer variety $K_{n-1}A \subset A^{[n]}$; see [Mea15, Lem. 6.2].

If $\omega_X$ is trivial, the canonical bundle of the product $\omega_X^n \cong \omega_X^{\otimes n}$ is trivial too. Similarly, for any abelian variety $A$, the canonical bundle of $N_{n-1}A$ is trivial.

There is the following fundamental difference depending on the parity of the dimensions of $X$ and $A$. Namely, in the even dimensional case also the equivariant canonical bundles $\omega_{S^nX} \in \text{Coh}(S^nX) \cong \text{Coh}_{S_n}(X^n)$ and $\omega_{K_{n-1}A} \in \text{Coh}(K_{n-1}A) \cong \text{Coh}_{S_n}(N_{n-1}A)$ are trivial.

But in the odd dimensional case we have

**Lemma 1.5.10.** If $\dim X$ and $\dim A$ are odd, we have $\omega_{S^nX} \cong \mathcal{O}_{X^n} \otimes \varpi$ and $\omega_{[N_{n-1}A/S_n]} \cong \mathcal{O}_{N_{n-1}A} \otimes \varpi$ where $\varpi$ denotes the sign representation of $S_n$, that is, the unique non-trivial character.

**Proof.** By Lemma 1.2.3 there are only two possible linearisations of $\omega_X^n$ so that the linearisation of the canonical bundle is determined by the $S_n$-action on the fibre $\omega_X^n(p)$ over a point $p = (x, \ldots, x)$ on the small diagonal. We denote by $\mathcal{C}^{\{1, \ldots, n\}}$ the permutation representation of $S_n$ and by $\varrho_n$ the standard representation, that is the quotient of $\mathcal{C}^{\{1, \ldots, n\}}$ by the one dimensional invariant subrepresentation, so that $\mathcal{C}^{\{1, \ldots, n\}} \cong \varrho_n \oplus \mathbb{C}$. We have $\det \mathcal{C}^{\{1, \ldots, n\}} \cong \det \varrho_n \cong \varpi$; see [FH91, Prop. 2.12]. Furthermore, $\Omega_X^n(p) \cong (\mathcal{C}^{\{1, \ldots, n\}})^{\otimes d}$ as $S_n$-representations where $d = \dim X$. Thus, $\omega_{X^n}(p) \cong \det \Omega_X^n(p) \cong \varpi^{\otimes d}$. Similarly, $\Omega_{N_{n-1}A}(p) \cong \varrho_n^{\otimes d}$ and thus also $\omega_{N_{n-1}A}(p) \cong \varpi^{\otimes d}$ for $d = \dim A$. □
Proposition 1.5.11. 1. Let $X$ be a smooth projective variety of odd dimension whose canonical bundle is trivial, ample, or anti-ample. Then, for $n \geq 2$, the symmetric quotient stack $S^n X = [X^n/\mathcal{S}_n]$ is not derived equivalent to any smooth projective variety.

2. Let $A$ be an abelian variety of odd dimension and $n \geq 2$. Then the generalised Kummer stack $K_{n-1} A = [N_{n-1} A/\mathcal{S}_n]$ is not derived equivalent to any smooth projective variety.

Proof. This follows from the Propositions 1.5.6 and 1.5.7 together with Lemma 1.5.10. For the case that $\omega_X$ is (anti-)ample, consider a point of the form $x = (y, y, x_3, \ldots, x_n) \in X^n$ with $x_3, \ldots, x_n$ pairwise distinct. Then the isotropy group of $x$ is $\mathcal{S}_2 \cong \mathbb{Z}/2\mathbb{Z}$ and acts by the non-trivial character on $\omega_X(x)$.

Corollary 1.5.12. Let $C$ be a smooth projective curve and $n \geq 2$. Then $S^n C$ is not derived equivalent to any smooth projective variety.

1.5.5 Kummer stacks

Let us consider the generalised Kummer stack in the case $n = 2$. Then $N_1 A \subset A^2$ is just the anti-diagonal and thus isomorphic to $A$. Under the isomorphism $N_1 A \cong A$ the non-trivial element of $\mathcal{S}_2$ acts by $i: A \to A$, $x \mapsto -x$. The global quotient stack $[A/i] \cong K_1 A$ is called the Kummer stack associated to $A$. Note that if $A$ is an abelian surface, there is the derived McKay correspondence $D^b([A]) \cong D^b([A/i]) \cong D^b(\text{K}(A))$ where $\text{K}(A)$ is the Kummer K3 surface associated to $A$. It is proven in [Ste07] that for every derived equivalence $D^b(A) \cong D^b(B)$ between abelian varieties there is a derived equivalence $D^b([A/i]) \cong D^b([B/i])$. In the case that $\dim A = \dim B$ is odd, also the converse holds:

Proposition 1.5.13. Let $A$ and $B$ be abelian varieties of odd dimension. Then $D^b([A/i]) \cong D^b([B/i])$ implies $D^b(A) \cong D^b(B)$.

Proof. As discussed in the previous subsection, the canonical bundle of $[A/i]$ is torsion of order 2 and the canonical cover is $\pi: A \to [A/i]$. Hence, this is just a special case of Corollary 1.4.5.

1.5.6 Enriques quotients of Hilbert schemes

Let $X$ be a smooth projective surface and $n \in \mathbb{N}$. Recall that there is the Bridgeland-King-Reid, Haiman (in the following abbreviated as BKR-H) equivalence

$$\Phi = FM_{\mathcal{O}_{I^n X}}: D^b(X^{[n]}) \cong D^b(S^n X)$$

where $I^n X = (X^{[n]} \times S^n X)^{\text{red}} \subset X^{[n]} \times X^n$ is the isospectral Hilbert scheme; see [BKR01] and [Hai01]. The fibre product is given by the quotient morphism $\pi: X^n \to S^n X := X^n/\mathcal{S}_n$ and the Hilbert–Chow morphism $HC: X^{[n]} \to S^n X$.

Any automorphism $\varphi \in \text{Aut}(X)$ induces automorphisms $\varphi^{[n]} \in \text{Aut}(X^{[n]})$, $\varphi^{x^n} \in \text{Aut}(X^n)$, and $S^n \varphi \in \text{Aut}(S^n X)$. The morphisms $HC$ and $\pi$ commute with these induced automorphisms. It follows that $I^n X$ is invariant under $\varphi^{[n]} \times \varphi^{x^n}$.

Let $\varphi$ be of finite order. Then $\varphi^{[n]}$ and $\varphi^{x^n}$ are of the same order and there is the isomorphism $\mu: \langle \varphi^{[n]} \rangle \cong \langle \varphi^{x^n} \rangle$ given by sending one generator to the other. Since $I^n X$ is invariant under $\varphi^{[n]} \times \varphi^{x^n}$, the object $\mathcal{O}_{I^n X}$ carries a canonical $\mu$-linearisation. Thus, Proposition 1.4.2 gives...
Corollary 1.5.14. The BKR-H equivalence $\Phi: \mathcal{D}^b(X^{[n]}) \xrightarrow{\cong} \mathcal{D}^b(S^nX)$ descends to an equivalence

$$\tilde{\Phi}: \mathcal{D}^b([X^{[n]}/\varphi^{[n]}]) \xrightarrow{\cong} \mathcal{D}^b([S^nX/\varphi^{\times n}])$$

between the derived categories of the quotient stacks. \hfill \Box

Another important functor occurring in the setup of Hilbert schemes of points on surfaces is the Fourier-Mukai transform $F_n = \text{FM}_{\mathcal{I}_S}: \mathcal{D}^b(X) \to \mathcal{D}^b(X^{[n]})$ along the universal ideal sheaf $\mathcal{I}_S$. Clearly, the universal family $\Xi \subset X \times X^{[n]}$ is invariant under $\varphi \times \varphi^{[n]}$. Hence, $\mathcal{I}_S \in \mathcal{D}^b(X \times X^{[n]})$ carries a canonical $\varphi \times \varphi^{[n]}$-linearisation. Therefore, we get

Corollary 1.5.15. For any surface $X$ the Fourier-Mukai transform $F_n = \text{FM}_{\mathcal{I}_S}$ with the structure sheaf of the universal family as kernel descends to a functor $\tilde{F}_n: \mathcal{D}^b([X/\varphi]) \to \mathcal{D}^b([X^{[n]}/\varphi^{[n]}])$. \hfill \Box

There are derived category versions of the Nakajima operators as defined in [7] given by Fourier-Mukai transforms

$$H_{\ell,n}: \mathcal{D}^b(X \times X^{[n]}) \cong \mathcal{D}^b_{\mathcal{E}_\ell}(X \times X^\ell) \to \mathcal{D}^b_{\mathcal{E}_{n+\ell}}(X^{n+\ell}) \cong \mathcal{D}^b(X^{[n+\ell]}).$$

Basically, they are constructed using pushforwards along the closed embeddings $\delta_I: X \times X^\ell \to X^{n+\ell}$ onto the partial diagonals $\Delta_I = \{(x_i, \ldots, x_n) | x_i = x_j \forall i, j \in I\}$ for $I \subset \{1, \ldots, n+\ell\}$ with $|I| = n$. The graph of $\delta_I$ is invariant under the morphism $\varphi \times \varphi^\times \times \varphi^{n+\ell}$. Hence, we get

Corollary 1.5.16. The Nakajima operators $H_{\ell,n}$ descend to

$$\tilde{H}_{\ell,n}: \mathcal{D}^b([X \times X^{[n]}/\varphi \times \varphi^\times]) \cong \mathcal{D}^b_{\mathcal{E}_\ell}([X \times X^\ell/\varphi \times \varphi^\times]) \to \mathcal{D}^b_{\mathcal{E}_{n+\ell}}([X^{n+\ell}/\varphi \times \varphi^{n+\ell}]) \cong \mathcal{D}^b([X^{[n+\ell]}/\varphi^{[n+\ell]}]).$$

For $n \geq \max\{\ell,2\}$ the $H_{\ell,n}$ are $\mathbb{P}^{n-1}$-functors satisfying (1.10); see [7]. Therefore, the descents $\tilde{H}_{\ell,n}$ are $\mathbb{P}^{n-1}$-functors too. \hfill \Box

Let $\tilde{X}$ be a K3 surface with a fixed point free involution $\tau: \tilde{X} \to \tilde{X}$ and let $X = \tilde{X}/\tau$ be the corresponding Enriques surface. For $n \in \mathbb{N}$ an odd number, the induced automorphism $S^n\tau: S^n\tilde{X} \to S^nX$ is a fixed point free involution. Hence, the same holds for $\tau^{[n]}$ and $\tau^{\times n}$. The quotient $X_{[n]} := \tilde{X}^{[n]}/\tau^{[n]}$ is the first example of a higher dimensional Enriques manifold in the sense of [OS11]. Note that $\tau^{\times n}$ commutes with the $\mathcal{G}_n$-action on $\tilde{X}^{[n]}$. Thus there is an induced $\mathcal{G}_n$-action on $X_n := \tilde{X}^{[n]}/\tau^{\times n}$. By the above discussion, we get an equivalence

$$\tilde{\Phi}: \mathcal{D}^b(X_{[n]}) \xrightarrow{\cong} \mathcal{D}^b_{\mathcal{E}_n}(X_n).$$

In fact, this itself is a special case of the BKR theorem. Indeed, one can show quite easily that $X_{[n]} \cong \text{Hilb}^n\tilde{X}(X_n)$.

In [Add16] it is proved that for a K3 surface $\tilde{X}$ the Fourier-Mukai transform $\tilde{F}_n: \mathcal{D}^b(\tilde{X}) \cong \mathcal{D}^b(\tilde{X}^{[n]})$ is a $\mathbb{P}^{n-1}$-functor. Therefore, the descent $\tilde{F}_n: \mathcal{D}^b(X) \to \mathcal{D}^b(X_{[n]})$ is again a $\mathbb{P}^{n-1}$-functor by Proposition 1.4.2.

Further examples of Enriques manifolds are given by quotients of generalised Kummer varieties; see [OS11], [BNS11]. Again, the quotients come from fixed point free automorphisms.
of finite order on the Kummer varieties which are naturally induced by automorphisms of the abelian surface. Therefore, the BKR-H equivalence descends to the corresponding quotients.

Furthermore, the Fourier-Mukai transform along the universal ideal sheaf is a \( \mathbb{P}^n \)-functor for generalised Kummer varieties; see [Mea15]. It descends to a \( \mathbb{P}^n \)-functor from the derived category of the hyperelliptic quotient of the abelian surface to the derived category of the Enriques quotient of the generalised Kummer variety.

### 1.5.7 Calabi-Yau covers of Hilbert schemes

Let \( X \) be a smooth projective surface. The line bundle \( L^\otimes n \in \operatorname{Pic}(X^n) \) carries a canonical \( \mathcal{O}_n \)-linearisation. There is also the induced line bundle \( L_n := \mathcal{H}^*((\pi_* L^\otimes n)^{\mathcal{O}_n}) \) (here we use notation from the previous subsection). Writing again \( \Phi \) for the BKR-H equivalence, we have

\[
\Phi \circ M_{L_n} \cong M_{L^{\otimes n}} \circ \Phi : \mathbb{D}^b(X^n) \to \mathbb{D}^b(\mathcal{O}_{n}) ;
\]  

see, for example, [Kru14, Lem. 9.2].

Let now \( L \) be of finite order. By (1.19) together with Proposition 1.4.3 and Remark 1.4.4 it follows that \( \Phi \) lifts to an equivalence

\[
\tilde{H}_{\ell,n} : \mathbb{D}^b((X \times X^\otimes [\ell])_{L^{\otimes L_{\ell}}}) \to \mathbb{D}^b(X^{[n+\ell]}) .
\]

Again, this equivalence is a special case of the BKR theorem as \( (X^n)_{L_n} \cong \operatorname{Hilb}^{\mathcal{O}_n}((X^n)_{L^{\otimes n}}) \).

Also, there is the relation \( M_{L^{\otimes (n+\ell)}} \circ H_{\ell,n} \cong H_{\ell,n} \circ M_{L^{\otimes L_{\ell}}} \). This shows that for \( \ell \geq 1 \) there is a lift to a \( \mathbb{P}^{n-1} \)-functor

\[
\tilde{H}_{\ell,n} : \mathbb{D}^b((X \times X^\otimes [\ell])_{L^{\otimes L_{\ell}}}) \to \mathbb{D}^b(X^{[n+\ell]}) .
\]

**Remark 1.5.17.** Probably the most interesting special case is when \( X \) is an Enriques surface and \( L = \omega_X \) is the canonical bundle. We have \( (\omega_X)_n = \omega_X^{[n]} \) and the canonical cover \( \operatorname{CY}_n(X) := X^{[n]} := X^{[n]}_X \) is a Calabi-Yau variety; see [Nie09, Prop. 1.6].

There is a second \( 2n \)-dimensional Calabi-Yau variety induced by an Enriques surface \( X \). Namely, the canonical cover \( \widetilde{\operatorname{CY}}_n(X) := \tilde{X}^n \) of the Cartesian product. Indeed, a smooth projective variety \( M \) is Calabi-Yau if and only if \( \mathcal{O}_M \) is a spherical object. Furthermore, \( \mathcal{O}_{X^n} = \mathcal{O}_{X}^{\otimes n} \) is exceptional and hence \( \mathcal{O}_{X^n} = \pi^* \mathcal{O}_X^{\otimes n} \) is spherical; see [ST01, Prop. 3.13] or [5, Rem. 3.11]. We have \( \mathbb{D}^b(\operatorname{CY}_n(X)) \cong \mathbb{D}^b(\widetilde{\operatorname{CY}}_n(X))^{\mathcal{O}_n} \).

More generally, let \( X_1, \ldots, X_m \) be smooth projective varieties with torsion canonical bundles of order 2 such that the structure sheaves \( \mathcal{O}_{X_i} \) are exceptional. Then \( \mathcal{O}_{X_1 \times \ldots \times X_m} \) is again exceptional and thus the canonical cover \( \widetilde{\operatorname{CY}}(X_1, \ldots, X_m) := X_1 \times \ldots \times X_m \) is Calabi-Yau.

Going the other way around, there are Calabi-Yau varieties with fixed point free involutions, so the quotient has canonical bundle of order 2 and its structure sheaf is an exceptional object, see [BNS11, Subsect. 4.3].

### 1.A Appendix: Necessary condition for lifts

In this section we will prove a converse of Proposition 1.4.3, namely
Proposition 1.A.1. Let $\pi_1: \tilde{X} \to X$ and $\pi_2: \tilde{Y} \to Y$ be Galois covers of smooth projective stacks with abelian groups of deck transformations $G$ and $G'$, respectively, and let $\Phi = \text{FM}_P: \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ be a fully faithful functor. Then, for $\Phi$ to lift to a fully faithful functor $\Psi: \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ it is necessary that the FM kernel $P \in \mathcal{D}^b(X \times Y)$ is $\mu$-linearisable for some isomorphism $\mu: \tilde{G} \simeq \tilde{G}'$.

Note that for $L \in \tilde{G} \subset \text{Pic}(X)$ the FM kernel of the autoequivalence $M_L$ is given by $\Delta_* \mathcal{L} \in \mathcal{D}^b(X \times X)$.

Lemma 1.A.2. For $L_1 \neq L_2 \in \tilde{G}$ we have $\text{Hom}_{\mathcal{D}^b(X \times X)}(\Delta_* L_1, \Delta_* L_2) = 0$.

Proof. We have $\text{Hom}(\Delta_* L_1, \Delta_* L_2) \cong \text{Hom}(L_1, L_2) \cong \Gamma(X, L_1^{-1} \otimes L_2)$. Since $L_1^{-1} \otimes L_2$ is a non-trivial torsion line bundle, it does not have non-zero global sections.

Lemma 1.A.3. Let $P \in \mathcal{D}^b(X \times Y)$ such that $\Phi = \text{FM}_P: \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is fully faithful. Then $\text{Hom}_{\mathcal{D}^b(X \times Y)}((L_1 \boxtimes \mathcal{O}) \otimes P, (L_2 \boxtimes \mathcal{O}) \otimes P) = 0$ for $L_1 \neq L_2 \in \tilde{G}$.

Proof. Since $\Phi \circ M_{L_1}$ is fully faithful, its FM kernel is unique up to isomorphism. Thus, by Lemma 1.2.7 we have $P \ast \Delta_* L_1 \cong (L_1 \boxtimes \mathcal{O}) \otimes P$. Since $\Phi$ is fully faithful, $P^R \ast P \cong O_{\Delta X}$. By (1.5) together with the previous lemma

$$\text{Hom}((L_1 \boxtimes \mathcal{O}) \otimes P, (L_2 \boxtimes \mathcal{O}) \otimes P) \cong \text{Hom}(P \ast \Delta_* L_1, P \ast \Delta_* L_1) \cong \text{Hom}(\Delta_* L_1, \Delta_* L_1) = 0.$$ 

Proof of Proposition 1.A.1. Let $\Psi: \mathcal{D}^b(\tilde{X}) \to \mathcal{D}^b(\tilde{Y})$ be a lift of $\Phi$ with FM kernel $Q \in \mathcal{D}^b(\tilde{X} \times \tilde{Y})$. By the definition of a lift, we have $\Psi \pi_1 \ast \pi_1^* \cong \pi_2 \ast \Psi \pi_1^* \cong \pi_2 \ast \pi_1^* \Phi$. The functor $\Phi \pi_1 \ast \pi_1^* \cong \oplus_{L \in \tilde{G}} \Phi \circ M_L$ satisfies (1.3) so that its FM kernel is unique up to isomorphism. By Lemma 1.2.7, the FM kernels of $\Phi \pi_1 \ast \pi_1^*$, $\pi_2 \ast \Psi \pi_1^*$, and $\pi_2 \ast \pi_2^* \Phi$ are $\oplus_{L \in \tilde{G}} (\mathcal{L} \boxtimes \mathcal{O}) \otimes P$, $Q := (\pi_1 \times \pi_2)_* Q$, and $\oplus_{L' \in \tilde{G}'} (\mathcal{O} \boxtimes L') \otimes P$, respectively. Hence,

$$\oplus_{L \in \tilde{G}} (\mathcal{L} \boxtimes \mathcal{O}) \otimes P \equiv Q \equiv \oplus_{L' \in \tilde{G}'} (\mathcal{O} \boxtimes L') \otimes P. \quad (1.20)$$

Since $\mathcal{D}^b(\tilde{X} \times \tilde{Y})$ is a Krull-Schmidt category, there is for every $L \in \tilde{G}$ a $\mu(L) \in \tilde{G}'$ such that $(\mathcal{L} \boxtimes \mathcal{O}) \otimes P \cong (\mathcal{O} \boxtimes \mu(L)) \otimes P$. By Lemma 1.A.3, the line bundle $\mu(L)$ is unique.

Because of the identification $\mathcal{D}^b(\tilde{X} \times \tilde{Y}) \cong \mathcal{D}^b_{\tilde{G} \times \tilde{G}'}(X \times Y)$, the object $Q$ carries a $\tilde{G} \times \tilde{G}'$-linearisation $\nu$. Now, the isomorphism $\nu_{(L^{-1}, O)}: Q \to (L^{-1} \boxtimes \mathcal{O}) \otimes Q$ restricts under (1.20) to an isomorphism

$$P \to (L^{-1} \boxtimes \mathcal{O}) \otimes (\mathcal{O} \boxtimes \mu(L)) \otimes P \cong (L^{-1} \boxtimes \mu(L)) \otimes P. \quad (1.21)$$

By the uniqueness of $\mu(L)$, the map $\mu$ is a homomorphism and (1.21) defines a $\mu$-linearisation of $P$.

References


Chapter 2

Derived categories of resolutions of cyclic quotient singularities

(Joint work with David Ploog and Paweł Sosna. arXiv:1701.01331.)

Abstract

For a cyclic group $G$ acting on a smooth variety $X$ with only one character occurring in the $G$-equivariant decomposition of the normal bundle of the fixed point locus, we study the derived categories of the orbifold $[X/G]$ and the blow-up resolution $\tilde{Y} \to X/G$.

Some results generalise known facts about $X = \mathbb{A}^n$ with diagonal $G$-action, while other results are new also in this basic case. In particular, if the codimension of the fixed point locus equals $|G|$, we study the induced tensor products under the equivalence $\mathbb{D}^b(\tilde{Y}) \cong \mathbb{D}^b([X/G])$ and give a 'flop-flop=twist' type formula. We also introduce candidates for general constructions of categorical crepant resolutions inside the derived category of a given geometric resolution of singularities and test these candidates on cyclic quotient singularities.

2.1 Introduction

For geometric, homological and other reasons, it has become commonplace to study the bounded derived category of a variety. One of the many intriguing aspects are connections, some of them conjectured, some of them proven, to birational geometry.

One expected phenomenon concerns a birational correspondence

\[
\begin{array}{ccc}
Z & \xrightarrow{q} & X \\
\parallel & & \parallel \\
\xleftarrow{p} & & \xrightarrow{\sim}
\end{array}
\]

of smooth varieties. Then we should have:

- A fully faithful embedding $\mathbb{D}^b(X) \hookrightarrow \mathbb{D}^b(X')$, if $q^*K_X \leq p^*K_{X'}$.
- A fully faithful embedding $\mathbb{D}^b(X') \hookrightarrow \mathbb{D}^b(X)$, if $q^*K_X \geq p^*K_{X'}$.
- An equivalence $\mathbb{D}^b(X') \cong \mathbb{D}^b(X)$, in the flop case $q^*K_X = p^*K_{X'}$. 

This is proven in many instances; see [BO95], [Bri02], [Kaw02], [Nam03].

Another very interesting aspect of derived categories is their occurrence in the context of the McKay correspondence. Here, one of the key expectations is that the derived category of a crepant resolution $\tilde{Y} \to X/G$ of a Gorenstein quotient variety is derived equivalent to the corresponding quotient orbifold: $\mathbb{D}^b(\tilde{Y}) \cong \mathbb{D}^b([X/G]) = \mathbb{D}^b_G(X)$. In [BKR01], this expectation is proven in many cases under the additional assumption that $\tilde{Y} \cong \text{Hilb}^G(X)$ is the fine moduli space of $G$-clusters on $X$. It is enlightening to view the derived McKay correspondence as an orbifold version of the conjecture on derived categories under birational correspondences described above; for more information on this point of view, see [Kaw16, Sect. 2], where the conjecture is called the DK-Hypothesis. Indeed, if we denote the universal family of $G$-clusters by $Z \subset \tilde{Y} \times X$, we have the following diagram of birational morphisms of orbifolds:

$$
\begin{array}{ccc}
[Z/G] & \xrightarrow{q} & [X/G] \\
\downarrow{p} & & \downarrow{\pi} \\
\tilde{Y} \leftarrow & \sim & X/G.
\end{array}
$$

Since the pullback of the canonical sheaf of $X/G$ under $\pi$ is the canonical sheaf of $[X/G]$, the condition that $\varrho$ is a crepant resolution amounts to saying that (2.1) is a flop of orbifolds.

In many situations, a crepant resolution of $X/G$ does not exist. However, given a resolution $\varrho: \tilde{Y} \to X$, the DK-Hypothesis still predicts the behavior of the categories $\mathbb{D}^b(\tilde{Y})$ and $\mathbb{D}^b_G(X)$ if $\varrho^*K_{X/G} \geq K_{\tilde{Y}}$ or $\varrho^*K_{X/G} \leq K_{\tilde{Y}}$. Another related idea is that, even though a crepant resolution does not exist in general, there should always be a categorical crepant resolution of $\mathbb{D}^b(X/G)$; see [Kuz08]. The hope is to find such a categorical resolution as an admissible subcategory of the derived category $\mathbb{D}^b(\tilde{Y})$ of a geometric resolution.

Besides dimensions 2 and 3, one of the most studied testing grounds for the above, and related, ideas is the isolated quotient singularity $\mathbb{A}^n/\mu_m$. Here, the cyclic group $\mu_m$ of order $m$ acts on the affine space by multiplication with a primitive $m$-th root of unity $\zeta$. In this paper, we consider the following straightforward generalisation of this set-up. Namely, let $X$ be a quasi-projective smooth complex variety acted upon by the finite cyclic group $\mu_m$. We assume that only 1 and $\mu_m$ occur as the isotropy groups of the action and write $S := \text{Fix}(\mu_m) \subset X$ for the fixed point locus. Fix a generator $g$ of $\mu_m$ and assume that $g$ acts on the normal bundle $N := N_{S/X}$ by multiplication with some fixed primitive $m$-th root of unity $\zeta$. Then the blow-up $\tilde{Y} \to X/\mu_m$ with center $S$ is a resolution of singularities; see Section 2.3 for further details.

The four particular cases we have in mind are

(a) $X = \mathbb{A}^n$ with the diagonal action of any $\mu_m$.

(b) $X = \mathbb{Z}^2$, where $Z$ is a smooth projective variety of arbitrary dimension, and $\mu_2 = \mathbb{S}_2$ acts by permuting the factors. Then $\tilde{Y} \cong \mathbb{Z}[2]$, the Hilbert scheme of two points.

(c) $X$ is an abelian variety, $\mu_2$ acts by $\pm 1$. In this case, $\tilde{Y}$ is known as the Kummer resolution.

(d) $X \to Y = X/\mu_m$ is a cyclic covering of a smooth variety $Y$, branched over a divisor. Here, $n = 1$ and $\tilde{X} = X$, $\tilde{Y} = Y$. This case has been studied in [KP17].
First, we prove the following result in Subsection 2.3.1. This is probably well-known to experts, but we could not find it in the literature. Write $G := \mu_m$.

**Proposition 2.1.1.** The resolution obtained by blowing up the fixed point locus in $X/G$ is isomorphic to the $G$-Hilbert scheme: $\tilde{Y} \cong \text{Hilb}^G(X)$.

We set $n := \text{codim}(S \hookrightarrow X)$ and find the following dichotomy, in accordance with the DK-Hypothesis. We keep the notation from diagram (2.1). In particular, for $n = m$, we obtain new instances of BKR-style derived equivalences between orbifold and resolution.

**Theorem 2.1.2.**

1. The functor $\Phi := p_* q^* : \mathcal{D}^b(\tilde{Y}) \to \mathcal{D}^b_G(X)$ is fully faithful for $m \geq n$ and an equivalence for $m = n$. For $m > n$, there is a semi-orthogonal decomposition of $\mathcal{D}^b_G(X)$ consisting of $\Phi(\mathcal{D}^b(\tilde{Y}))$ and $m - n$ pieces equivalent to $\mathcal{D}^b(S)$.

2. The functor $\Psi := q_* p^* : \mathcal{D}^b_G(X) \to \mathcal{D}^b(\tilde{Y})$ is fully faithful for $n \geq m$ and an equivalence for $n = m$. For $n > m$, there is a semi-orthogonal decomposition of $\mathcal{D}^b(\tilde{Y})$ consisting of $\Psi(\mathcal{D}^b_G(X))$ and $n - m$ pieces equivalent to $\mathcal{D}^b(S)$.

For a more exact statement with an explicit description of the embeddings of the $\mathcal{D}^b(S)$ components into $\mathcal{D}^b(\tilde{Y})$ and $\mathcal{D}^b_G(X)$, see Section 2.4. In particular, for $m > n$, the push-forward $a_* : \mathcal{D}^b(S) \to \mathcal{D}^b_G(X)$ along the embedding $a : S \hookrightarrow X$ of the fixed point divisor is fully faithful.

In the basic affine case (a), the result of the theorem is also stated in [Kaw16, Ex. 4]. Proofs, in this basic case, are given in [Abu16, Sect. 4] for $n \geq m$ and in [IU15] for $n = 2$. If $n = 1$, the quotient is already smooth and we have $\tilde{Y} = X/G$ — here the semi-orthogonal decomposition categorifies the natural decomposition of the orbifold cohomology; compare [PV15]. The $n = 1$ case is also proven in [Lim16, Thm. 3.3.2].

We study the case $m = n$, where $\Phi$ and $\Psi$ are equivalences, in more detail. On both sides of the equivalence, we have distinguished line bundles. The line bundle $\mathcal{O}_{\tilde{Y}}(Z)$ on $\tilde{Y}$, corresponding to the exceptional divisor, admits an $m$-th root $\mathcal{L}$. On $[X/G]$, there are twists of the trivial line bundle by the group characters $\mathcal{O}_X \otimes \chi^i$. For $i = -m + 1, \ldots, -1, 0$, we have $\Psi(\mathcal{O}_X \otimes \chi^i) \cong \mathcal{L}^i$. Furthermore, we see that the functors $\mathcal{D}^b(S) \to \mathcal{D}^b(\tilde{Y})$ and $\mathcal{D}^b(S) \to \mathcal{D}^b_G(X)$, which define fully faithful embeddings in the $n > m$ and $m < n$ cases, respectively, become spherical for $m = n$ and hence induce twist autoequivalences; see Subsection 2.2.8 for details on spherical functors and twists. We show that the tensor products by the distinguished line bundles correspond to the spherical twists under the equivalences $\Psi$ and $\Phi$. In particular, one part of Theorem 2.4.26 is the following formula.

**Theorem 2.1.3.** There is an isomorphism $\Psi^{-1}(\Psi(\_ \otimes \_ \otimes \mathcal{L}^{-1}) \cong T^{-1}_a(\_ \otimes \chi^{-1})$ of autoequivalences of $\mathcal{D}^b_G(X)$ where the inverse spherical twist $T^{-1}_a$ is defined by the exact triangle of functors

\[ T^{-1}_a \to \text{id} \to a_*(a^*(\_ \otimes G)) \to . \]

The tensor powers of the line bundle $\mathcal{L}$ form a strong generator of $\mathcal{D}^b(\tilde{Y})$, thus Theorem 2.1.3, at least theoretically, completely describes the tensor product

\[ \_ \otimes \_ := \Psi^{-1}(\Psi(\_ \otimes \_)) : \mathcal{D}^b_G(X) \times \mathcal{D}^b_G(X) \to \mathcal{D}^b_G(X) \]
induced by $\Psi$ on $\operatorname{D}^b_G(X)$. There is related unpublished work on induced tensor products under the McKay correspondence in dimensions 2 and 3 by T. Abdelgadir, A. Craw, J. Karmazyn, and A. King. In Corollary 2.4.27, we also get a formula which can be seen as a stacky instance of the 'flop-flop = twist' principle as discussed in [ADM15].

In Section 2.5, we introduce a general candidate for a weakly crepant categorical resolution (see [Kuz08] or Subsection 2.5.1 for this notion), namely the weakly crepant neighbourhood $\operatorname{WC}(g) \subset \operatorname{D}^b(Y)$, inside the derived category of a given resolution $g: Y \to Y$ of a rational Gorenstein variety $Y$. The idea is pretty simple: by Grothendieck duality, there is a canonical section $s : \mathcal{O}_Y \to \mathcal{O}_g$ of the relative dualising sheaf, and this induces a morphism of Fourier–Mukai transforms $t := \varrho_s(\_ \otimes s) : \varrho_\ast \to \varrho_\ast$.

Set $\varrho_+ := \operatorname{cone}(t)$ and $\operatorname{WC}(g) := \ker(\varrho_+)$. Then, by the very construction, we have $\varrho_+|\operatorname{WC}(g) \cong \varrho_!|\operatorname{WC}(g)$ which amounts to the notion of categorical weak crepancy. The only thing open to ensure that $\operatorname{WC}(g)$ is a categorical weakly crepant resolution is whether it is actually a smooth category; this holds as soon as it is an admissible subcategory of $\operatorname{D}^b(Y)$ which means that its inclusion has adjoints. We prove that, in the Gorenstein case $m | n$ of our set-up of cyclic quotients, $\operatorname{WC}(g) \subset \operatorname{D}^b(Y)$ is an admissible subcategory; see Theorem 2.5.4.

In Subsection 2.5.4, we observe that there are various weakly crepant resolutions inside $\operatorname{D}^b(Y)$. However, a strongly crepant categorical resolution inside $\operatorname{D}^b(Y)$ is unique, as we show in Proposition 2.5.9. Our concept of weakly crepant neighbourhoods was motivated by the idea that some non-CY objects possess ‘CY neighbourhoods’ (a construction akin to the spherical subcategories of spherelike objects in [HKP16]), i.e. full subcategories in which they become Calabi–Yau. This relationship is explained in Subsection 2.5.5.

In the final Section 2.6, we construct Bridgeland stability conditions on Kummer threefolds as an application of our results; see Corollary 2.6.2.

**Conventions.** We work over the complex numbers. All functors are assumed to be derived. We write $H^i(E)$ for the $i$-th cohomology object of a complex $E \in \operatorname{D}^b(Z)$ and $H^n(E)$ for the complex $\oplus_i H^i(Z, E)[-i]$. If a functor $\Phi$ has a left/right adjoint, they are denoted $\Phi^L$, $\Phi^R$.

There are a number of spaces, maps and functors repeatedly used in this text. For the convenience of the reader, we collect our notation at the very end of this article, on page 98.

**Acknowledgements.** It is a pleasure to thank Tarig Abdelgadir, Martin Kalck, Sönke Rol lenske and Evgeny Shinder for comments and discussions.

### 2.2 Preliminaries

#### 2.2.1 Fourier–Mukai transforms and kernels

Recall that given an object $\mathcal{E}$ in $\operatorname{D}^b(Z \times Z')$, where $Z$ and $Z'$ are smooth and projective, we get an exact functor $\operatorname{D}^b(Z) \to \operatorname{D}^b(Z')$, $F \mapsto p_{Z'}^!(\mathcal{E} \otimes p_{Z'}^*F)$. Such a functor, denoted by $\operatorname{FM}_\mathcal{E}$, is called a Fourier–Mukai transform (or FM transform) and $\mathcal{E}$ is its kernel. See [Huy06] for a thorough introduction to FM transforms. For example, if $\Delta : Z \to Z \times Z$ is the diagonal map and $\mathcal{L}$ is in $\operatorname{Pic}(Z)$, then $\operatorname{FM}_{\Delta_*\mathcal{L}}(F) = F \otimes \mathcal{L}$. In particular, $\operatorname{FM}_{\operatorname{O}_\Delta}$ is the identity functor.

**Convention.** We will write $M_\mathcal{E}$ for the functor $\operatorname{FM}_{\Delta_*\mathcal{E}}$.

The calculus of FM transforms is, of course, not restricted to smooth and projective varieties. Note that $f_*$ maps $\operatorname{D}^b(Z)$ to $\operatorname{D}^b(Z')$ as soon as $f : Z \to Z'$ is proper. In order to
be able to control the tensor product and pullbacks, one can restrict to perfect complexes.
Recall that a complex of sheaves on a quasi-projective variety \( Z \) is called \textit{perfect} if it is locally quasi-isomorphic to a bounded complex of locally free sheaves. The triangulated category of perfect complexes on \( Z \) is denoted by \( D^\text{perf}(Z) \). It is a full subcategory of \( D^b(Z) \). These two categories coincide if and only if \( Z \) is smooth.

We will sometimes take cones of morphisms between FM transforms. Of course, one needs to make sure that these cones actually exist. Luckily, if one works with FM transforms, this is not a problem, because the maps between the functors come from the underlying kernels and everything works out, even for (reasonable) schemes which are not necessarily smooth and projective; see [AL12].

### 2.2.2 Group actions and derived categories

Let \( G \) be a finite group acting on a smooth variety \( X \). Recall that a \( G \)-equivariant coherent sheaf is a pair \((F, \lambda_g)\), where \( F \in \text{Coh}(X) \) and \( \lambda_g : F \xrightarrow{\sim} g^*F \) are isomorphisms satisfying a cocycle condition. The category of \( G \)-equivariant coherent sheaves on \( X \) is denoted by \( \text{Coh}^G(X) \). It is an abelian category. The \textit{equivariant derived category}, denoted by \( D^b(G)(X) \), is defined as \( D^b(\text{Coh}^G(X)) \), see, for example, [Plo07] for details. Recall that for every subgroup \( G' \subset G \) the restriction functor \( \text{Res} : D^b_G(X) \to D^b_{G'}(X) \) has the induction functor \( \text{Ind} : D^b_{G'}(X) \to D^b_G(X) \) as a left and right adjoint (see e.g. [Plo07, Sect. 1.4]). It is given for \( F \in D^b(Z) \) by

\[
\text{Ind}(F) = \bigoplus_{[g] \in G/G} g^*F
\]

with the \( G \)-linearisation given by the \( G' \)-linearisation of \( F \) together with appropriate permutations of the summands.

If \( G \) acts trivially on \( X \), there is also the functor \( \text{triv} : D^b(X) \to D^b_G(X) \) which equips an object with the trivial \( G \)-linearisation. Its left and right adjoint is the functor \((\_)^G : D^b_G(X) \to D^b(X) \) of invariants.

Given an equivariant morphism \( f : X \to X' \) between varieties endowed with \( G \)-actions, there are equivariant pushforward and pullback functors, see, for example, [Plo07, Sect. 1.3] for details. We will sometimes write \( f^G_* \) for \((\_)^G \circ f_* \). It is also well-known that the category \( D^b_G(X) \) has a tensor product and the usual formulas, e.g. the adjunction formula, hold in the equivariant setting.

Finally, we need to recall that a character \( \kappa \) of \( G \) acts on the equivariant category by twisting the linearisation isomorphisms with \( \kappa \). If \( F \in D^b_G(X) \), we will write \( F \otimes \kappa \) for this operation. We will tacitly use that twisting by characters commutes with the equivariant pushforward and pullback functors along \( G \)-equivariant maps.

### 2.2.3 Semi-orthogonal decompositions

References for the following facts are, for example, [Bon89] and [BO95].

Let \( \mathcal{T} \) be a Hom-finite triangulated category. A \textit{semi-orthogonal decomposition} of \( \mathcal{T} \) is a sequence of full triangulated subcategories \( A_1, \ldots, A_m \) such that (a) if \( A_i \in A_i \) and \( A_j \in A_j \), then \( \text{Hom}(A_i, A_j[l]) = 0 \) for \( i > j \) and all \( l \), and (b) the \( A_i \) generate \( \mathcal{T} \), that is, the smallest triangulated subcategory of \( \mathcal{T} \) containing all the \( A_i \) is already \( \mathcal{T} \). We write \( \mathcal{T} = \langle A_1, \ldots, A_m \rangle \).

If \( m = 2 \), these conditions boil down to the existence of a functorial exact triangle \( A_2 \to T \to A_1 \) for any object \( T \in \mathcal{T} \).
A subcategory $\mathcal{A}$ of $\mathcal{T}$ is right admissible if the embedding functor $\iota$ has a right adjoint $\iota^R$, left admissible if $\iota$ has a left adjoint $\iota^L$, and admissible if it is left and right admissible.

Given any triangulated subcategory $\mathcal{A}$ of $\mathcal{T}$, the full subcategory $\mathcal{A}^\perp \subseteq \mathcal{T}$ consists of objects $T$ such that $\text{Hom}(A,T[k]) = 0$ for all $A \in \mathcal{A}$ and all $k \in \mathbb{Z}$. If $\mathcal{A}$ is right admissible, then $\mathcal{T} = \langle \mathcal{A}^\perp, \mathcal{A} \rangle$ is a semi-orthogonal decomposition. Similarly, $\mathcal{T} = \langle \mathcal{A}, \perp \mathcal{A} \rangle$ is a semi-orthogonal decomposition if $\mathcal{A}$ is left admissible, where $\perp \mathcal{A}$ is defined in the obvious way.

Examples typically arise from so-called exceptional objects. Recall that an object $E \in \text{D}^b(Z)$ (or any $\mathbb{C}$-linear triangulated category) is called exceptional if $\text{Hom}(E,E) = \mathbb{C}$ and $\text{Hom}(E,E[k]) = 0$ for all $k \neq 0$. The smallest triangulated subcategory containing $E$ is then equivalent to $\text{D}^b(\text{Spec}(\mathbb{C}))$ and this category, by abuse of notation again denoted by $E$, is admissible, leading to a semi-orthogonal decomposition $\text{D}^b(Z) = \langle E, E \rangle$. A sequence of objects $E_1,\ldots,E_n$ is called an exceptional collection if $\text{D}^b(Z) = ((E_1,\ldots,E_n)^\perp, E_1,\ldots,E_n)$ and all $E_i$ are exceptional. The collection is called full if $(E_1,\ldots,E_n)^\perp = 0$.

Note that any fully faithful FM transform $\Phi: \text{D}^b(X) \to \text{D}^b(X')$ gives a semi-orthogonal decomposition $\text{D}^b(X') = (\Phi(\text{D}^b(X))^\perp, \Phi(\text{D}^b(X)))$, because any FM transform has a right and a left adjoint, see [Huy06, Prop. 5.9].

### 2.2.4 Dual semi-orthogonal decompositions

Let $\mathcal{T}$ be a triangulated category together with a semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{A}_1,\ldots,\mathcal{A}_n \rangle$. Then there is the left-dual semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{B}_n,\ldots,\mathcal{B}_1 \rangle$ given by $\mathcal{B}_i := \langle \mathcal{A}_1,\ldots,\mathcal{A}_{i-1},\mathcal{A}_{i+1},\ldots,\mathcal{A}_n \rangle^\perp$. There is also a right-dual decomposition but we will always use the left-dual and refer to it simply as the dual semi-orthogonal decomposition. We summarise the properties of the dual semi-orthogonal decomposition needed later on in the following

**Lemma 2.2.1.** Let $\mathcal{T} = \langle \mathcal{A}_1,\ldots,\mathcal{A}_n \rangle$ be a semi-orthogonal decomposition with dual semi-orthogonal decomposition $\mathcal{T} = \langle \mathcal{B}_n,\ldots,\mathcal{B}_1 \rangle$.

1. $\langle \mathcal{A}_1,\ldots,\mathcal{A}_r \rangle = \langle \mathcal{B}_r,\ldots,\mathcal{B}_1 \rangle$ and $\langle \mathcal{A}_1,\ldots,\mathcal{A}_r \rangle^\perp = \langle \mathcal{B}_n,\ldots,\mathcal{B}_{r+1} \rangle^\perp$ for $1 \leq r \leq n$.

2. If $\langle \mathcal{A}_1,\ldots,\mathcal{A}_n \rangle$ is given by an exceptional collection, i.e. $\mathcal{A}_i = \langle E_i \rangle$, then its dual is also given by an exceptional collection $\mathcal{B}_i = \langle F_i \rangle$ such that $\text{Hom}^*(E_i,F_j) = \delta_{ij}\mathbb{C}[0]$.

**Proof.** Part (i) is [Efi14, Prop. 2.7(i)]. Part (ii) is then clear. \hfill \Box

An important classical example is the following

**Lemma 2.2.2.** There are dual semi-orthogonal decompositions

$$\text{D}^b(\mathbb{P}^{n-1}) = \langle \mathcal{O}, \mathcal{O}(1),\ldots,\mathcal{O}(n-1) \rangle,$$

$$\text{D}^b(\mathbb{P}^{n-1}) = \langle \Omega^{n-1}(n-1)[n-1],\ldots,\Omega^1(1)[1],\mathcal{O} \rangle.$$

**Proof.** The fact that both sequences are indeed full goes back to Beilinson, see [Huy06, Sect. 8.3] for an account. The fact that they are dual is classical and follows by a direct computation, for instance using [BS10, Lem. 2.5]. \hfill \Box

The following relative version is the example of dual semi-orthogonal decompositions which we will need throughout the text.
Lemma 2.2.3. Let $\nu: Z \to S$ be a $\mathbb{P}^{n-1}$-bundle. There is the semi-orthogonal decomposition
\[ D^b(Z) = \langle \nu^* D^b(S), \nu^* D^b(S) \otimes O_\nu(1), \ldots, \nu^* D^b(S) \otimes O_\nu(n-1) \rangle \]
whose dual decomposition is given by
\[ D^b(Z) = \langle \nu^* D^b(S) \otimes \Omega^m_\nu(n-1), \ldots, \nu^* D^b(S) \otimes \Omega^m_\nu(1), \nu^* D^b(S) \rangle . \]

Proof. Part (i) is [Orl92, Thm. 2.6]. Part (ii) follows from Lemma 2.2.2. \qed

The following consequence will be used in Subsection 2.4.5.

Corollary 2.2.4. If $m < n$, there is the equality of subcategories of $D^b(Z)$
\[ \langle \nu^* D^b(S) \otimes O_\nu(m-n), \ldots, \nu^* D^b(S) \otimes O_\nu(-1) \rangle = \langle \nu^* D^b(S) \otimes \Omega^m_\nu(n-1), \ldots, \nu^* D^b(S) \otimes \Omega^m_\nu(1) \rangle . \]

Proof. Applying Lemma 2.2.1(i) to the dual decompositions of Lemma 2.2.3 gives the equalities
\[ \langle \nu^* D^b(S) \otimes \Omega^m_\nu(n-1), \ldots, \nu^* D^b(S) \otimes \Omega^m_\nu(1) \rangle = \langle \nu^* D^b(S), \ldots, \nu^* D^b(S) \otimes O_\nu(m-1) \rangle \]
\[ = \langle \nu^* D^b(S), \ldots, \nu^* D^b(S) \otimes O_\nu(m-n) \rangle \]
\[ = \langle \nu^* D^b(S) \otimes O_\nu(m-n), \ldots, \nu^* D^b(S) \otimes O_\nu(-1) \rangle . \] \qed

2.2.5 Linear functors and linear semi-orthogonal decompositions

Let $\mathcal{T}$ be a tensor triangulated category, i.e. a triangulated category with a compatible symmetric monoidal structure. Moreover, let $\mathcal{X}$ be a triangulated module category over $\mathcal{T}$, i.e. there is an exact functor $\pi^*: \mathcal{T} \to \mathcal{X}$ and a tensor product $\otimes: \mathcal{T} \times \mathcal{X} \to \mathcal{X}$, that is an assignment $\pi^*(A) \otimes E$ functorial in $A \in \mathcal{T}$ and $E \in \mathcal{X}$.

We will take $\mathcal{T} = D^b_G(Y)$ for some variety $Y$ with an action by a finite group $G$. Note that $D^b_G(Y)$ has a (derived) tensor product, and it is compatible with $G$-linearisations.

For $\mathcal{X}$, we have several cases in mind. If $X$ is a smooth $G$-variety with a $G$-equivariant morphism $\pi: X \to Y$, then we take $\mathcal{X} = D^b_G(X) = D^b_G(X)$; this is a tensor triangulated category itself and $\pi^*$ preserves these structures.

If $\Lambda$ is an $\mathcal{O}_Y$-algebra, then let $\mathcal{X} = D^b(\Lambda)$ be the bounded derived category of finitely generated right $\Lambda$-modules with $\pi^*(A) = A \otimes O_Y \Lambda$ and $\pi^*(A) \otimes E = \pi^*(A) \otimes O_Y \Lambda \otimes E = \pi^*(A) \otimes O_Y E \in \mathcal{X}$. Note that if $\Lambda$ is not commutative, then $\mathcal{X}$ is not a tensor category.

We say that a full triangulated subcategory $\mathcal{A} \subset \mathcal{X}$ is $\mathcal{T}$-linear (since in our cases we have $\mathcal{T} = D^b_G(Y)$ we will also speak of $Y$-linearity) if
\[ \pi^*(A) \otimes E \in \mathcal{A} \quad \text{for all } A \in \mathcal{T} \text{ and } E \in \mathcal{X}. \]
We say that a semi-orthogonal decomposition $\mathcal{X} = \langle A_1, \ldots, A_n \rangle$ is $\mathcal{T}$-linear, if all the $A_i$ are $\mathcal{T}$-linear subcategories.

We call a class of objects $\mathcal{S} \subset \mathcal{X}$ (left/right) spanning over $\mathcal{T}$ if $\pi^* \mathcal{T} \otimes \mathcal{S}$ is a (left/right) spanning class of $\mathcal{X}$ in the non-relative sense. Recall that a subset $\mathcal{C} \subset \mathcal{X}$ is generating if $\mathcal{X} = \langle \mathcal{C} \rangle$ is the smallest triangulated category closed under direct summands containing $\mathcal{C}$. The subset $\mathcal{C} \subset \mathcal{X}$ is called generating over $\mathcal{T}$ if $\mathcal{C} \otimes \pi^* \mathcal{T}$ generates $D^b(\mathcal{X})$. 

69
Let $\mathcal{X}'$ be a further tensor triangulated category together with a tensor triangulated functor $\pi'^*: \mathcal{T} \to \mathcal{X}'$. We say that an exact functor $\Phi: \mathcal{X} \to \mathcal{X}'$ is $\mathcal{T}$-linear if there are functorial isomorphisms

$$\Phi(\pi^*(A) \otimes E) \cong \pi'^*(A) \otimes \Phi(E) \quad \text{for all } A \in \mathcal{T} \text{ and } E \in \mathcal{X}.$$

The verification of the following lemma is straight-forward.

**Lemma 2.2.5.**

1. If $\Phi: \mathcal{X} \to \mathcal{X}'$ is $\mathcal{T}$-linear, then $\Phi(\mathcal{X})$ is a $\mathcal{T}$-linear subcategory of $\mathcal{X}'$.

2. Let $A \subset D^b(\mathcal{Y})$ be a $\mathcal{T}$-linear (left/right) admissible subcategory. Then the essential image of $A$ is $D^b(\mathcal{Y})$ if and only if $A$ contains a (left/right) spanning class over $\mathcal{T}$.

For the following, we consider the case that $\mathcal{X} = D^b(X)$ for some smooth variety $X$ together with a proper morphism $\pi: X \to Y$.

**Lemma 2.2.6.** Let $A, B \subset D^b(\mathcal{X})$ be $\mathcal{Y}$-linear full subcategories. Then

$$A \subset B^\perp \iff \pi_* \operatorname{Hom}(B, A) = 0 \quad \forall A \in A, B \in B.$$

**Proof.** The direction $\iff$ follows immediately from $\operatorname{Hom}^*(B, A) \cong \Gamma(Y, \pi_* \operatorname{Hom}(B, A))$; recall that all our functors are the derived versions.

Conversely, assume that there are $A \in A$ and $B \in B$ such that $\pi_* \operatorname{Hom}(B, A) \neq 0$. Since $D^\text{perf}(\mathcal{Y})$ spans $D(QCoh(\mathcal{Y}))$, this implies that there is an $E \in D^\text{perf}(\mathcal{Y})$ such that

$$0 \neq \operatorname{Hom}^*(E, \pi_* \operatorname{Hom}(B, A)) \cong \Gamma(Y, \pi_* \operatorname{Hom}(B, A) \otimes E^\vee) \cong \Gamma(Y, \pi_* (\operatorname{Hom}(B, A) \otimes \pi^* E^\vee))$$

$$\cong \Gamma(Y, \pi_* \operatorname{Hom}(B \otimes \pi^* E, A))$$

$$\cong \operatorname{Hom}^*(B \otimes \pi^* E, A).$$

By the $\mathcal{Y}$-linearity, we have $B \otimes \pi^* E \in B$ and hence $A \not\subset B^\perp$. \hfill \Box

### 2.2.6 Relative Fourier–Mukai transforms

Let $\pi: X \to Y$ and $\pi': X' \to Y$ be proper morphisms of varieties with $X$ and $X'$ being smooth. We denote the closed embedding of the fibre product into the product by $i: X \times_Y X' \hookrightarrow X \times X'$.

We call $\Phi: D^b(X) \to D^b(X')$ a relative FM transform if $\Phi = FM_{\pi, p}$ for some object $\mathcal{P} \in D^b(X \times_Y X)$. It is a standard computation that a relative FM transform is linear over $Y$, with respect to the pullbacks $\pi^*$ and $\pi'^*$. Furthermore, we have $\Phi \cong p_*(q^*(\_ \otimes \mathcal{P}))$ where $p$ and $q$ are the projections of the fibre diagram

$$\begin{array}{ccc}
X \times_Y X' & \xrightarrow{q} & X' \\
\downarrow \pi & & \downarrow \pi' \\
X & \xleftarrow{p} & Y
\end{array} \quad \text{(2.3)}$$

The right adjoint of $\Phi$ is given by $\Phi^R := q_*(p'(\_ \otimes \mathcal{P}^\vee)): D^b(X') \to D^b(X)$. We also have the following slightly stronger statement which one could call relative adjointness.
Lemma 2.2.7. For $E \in D^b(X)$ and $F \in D^b(X')$, there are functorial isomorphisms

$$\pi'_* \text{Hom}(\Phi(E), F) \cong \pi_* \text{Hom}(E, \Phi^R(F)).$$

Proof. This follows by Grothendieck duality, commutativity of (2.3), and projection formula:

$$\pi'_* \text{Hom}(\Phi(E), F) \cong \pi'_* \text{Hom}(p_*(q^*E \otimes \mathcal{P}), F) \cong \pi'_* p_* \text{Hom}(q^*E \otimes \mathcal{P}, p^!F)$$

$$\cong \pi_* q_* \text{Hom}(q^*E, p^!F \otimes \mathcal{P}^\vee) \cong \pi_* \text{Hom}(E, q_*(p^!F \otimes \mathcal{P}^\vee)) \cong \pi_* \text{Hom}(E, \Phi^R(F)).$$

For $E, F \in D^b(X)$, using the isomorphism of the previous lemma, we can construct a natural morphism

$$\tilde{\Phi} : \pi_* \text{Hom}(E, F) \to \pi'_* \text{Hom}(\Phi(E), \Phi(F))$$

as the composition

$$\tilde{\Phi} : \pi_* \text{Hom}(E, F) \to \pi'_* \text{Hom}(E, \Phi^R(F)) \equiv \pi'_* \text{Hom}(\Phi(E), \Phi(F))$$

where the first morphism is induced by the unit of adjunction $F \to \Phi^R \Phi(F)$. Note that taking global sections gives back the functor $\Phi$ on morphisms, i.e. $\Phi = \Gamma(Y, \tilde{\Phi})$ as maps

$$\text{Hom}^*(E, F) \cong \Gamma(Y, \pi_* \text{Hom}(E, F)) \to \Gamma(Y, \pi'_* \text{Hom}(\Phi(E), \Phi(F))) \cong \text{Hom}^*(\Phi(E), \Phi(F)).$$

More generally, $\Phi$ induces functors for open subsets $U \subseteq Y$,

$$\Phi_U : D^b(W) \to D^b(W'),$$

given by restricting the FM kernel $\iota_* \mathcal{P}$ to $W \times W'$ and we have $\Phi_U = \Gamma(U, \tilde{\Phi})$. From this we see that $\Phi$ is compatible with composition which means that the following diagram, for $E, F, G \in D^b(X)$, commutes

$$\begin{array}{c}
\pi_* \text{Hom}(F, G) \otimes \pi_* \text{Hom}(E, F) \\
\downarrow \pi'_* \text{Hom}(\Phi(F), \Phi(G)) \otimes \pi'_* \text{Hom}(\Phi(E), \Phi(F)) \\
\pi'_* \text{Hom}(\Phi(F), \Phi(G)) \\
\end{array}$$

$$\begin{array}{c}
\pi'_* \text{Hom}(\Phi(F), \Phi(G)) \otimes \pi'_* \text{Hom}(\Phi(E), \Phi(F)) \\
\downarrow \Phi \\
\pi'_* \text{Hom}(\Phi(F), \Phi(G)) \\
\end{array}$$

(2.5)

2.2.7 Relative tilting bundles

Let $\pi : X \to Y$ be a proper morphism of varieties and let $X$ be smooth. We say that $V \in D^b(X)$ is a relative tilting object if $\Lambda_V := \Lambda := \pi_* \text{Hom}(V, V)$ is cohomologically concentrated in degree 0 and $V$ is a spanning class over $Y$. Note that $\Lambda$ is an $\mathcal{O}_Y$-algebra. We denote the bounded derived category of coherent right modules over $\Lambda$ by $D^b(\Lambda)$. It is a triangulated module category over $D^\text{perf}(Y)$ via $\pi^* A = A \otimes_{\mathcal{O}_Y} \Lambda$, and $\Lambda$ is a relative generator. In particular, for $A \in D^b(X)$ and $M \in D^b(\Lambda)$, the tensor product $A \otimes M$ is over the base $\mathcal{O}_Y$. We get a relative tilting equivalence:

Proposition 2.2.8. Let $V \in D^b(X)$ be a relative tilting object over $Y$. Then $V$ generates $D^b(X)$ over $Y$, and the following functor defines a $Y$-linear exact equivalence:

$$t_V := \pi_* \text{Hom}(V, -) : D^b(X) \to D^b(\Lambda).$$
Proof. The $Y$-linearity of $t_V$ is due to the projection formula

$$t_V(\pi^* A \otimes E) = \pi_*(\pi^* A \otimes \text{Hom}(V, E)) \cong A \otimes \pi_* \text{Hom}(V, E) = A \otimes t_V(E).$$

Consider the restricted functor $t'_V : \mathcal{V} := (V \otimes \pi^* \mathcal{D}^{\text{perf}}(Y)) \to \mathcal{D}^b(\Lambda)$. We show that $t'_V$ is fully faithful, using the adjunctions $\pi^* \dashv \pi_*$ and $\_ \otimes \mathcal{O}_Y \Lambda \dashv \_$. For where $\text{For} : \mathcal{D}^b(\Lambda) \to \mathcal{D}^b(Y)$ is scalar restriction, the projection formula, and the $Y$-linearity of $t'_V$:

$$\text{Hom}_{\mathcal{O}_X}(\pi^* A \otimes V, \pi^* B \otimes V) \cong \text{Hom}_{\mathcal{O}_Y}(A, \pi_*(\pi^* B \otimes \text{Hom}(V, V)))$$
$$\cong \text{Hom}_{\mathcal{O}_Y}(A, \pi_*(\pi^* B \otimes \text{Hom}(V, V)))$$
$$\cong \text{Hom}_{\Lambda}(A \otimes \Lambda, B \otimes \Lambda)$$
$$\cong \text{Hom}_{\Lambda}(t'_V(\pi^* A \otimes V), t'_V(\pi^* B \otimes V))$$

Since objects of type $\pi^* A \otimes V$ generate $\mathcal{V}$, this shows that $t'_V$ is fully faithful. We have $t'_V(V) = \Lambda$. Since $\Lambda$ is a relative generator, hence a relative spanning class, of $\mathcal{D}^b(\Lambda)$, we get an equivalence $\mathcal{V} \cong \mathcal{D}^b(\Lambda)$; see Lemma 2.2.5.

We now claim that the inclusion $\mathcal{V} \hookrightarrow \mathcal{D}^b(X)$ has a right adjoint, namely

$$t'^{-1}_V : \mathcal{D}^b(X) \to \mathcal{D}^b(\Lambda) \to \mathcal{V}.$$ 

For this, take $A \in \mathcal{D}^{\text{perf}}(Y)$, $F \in \mathcal{D}^b(X)$ and compute

$$\text{Hom}_{\mathcal{O}_X}(\pi^* A \otimes V, F) \cong \text{Hom}_{\mathcal{O}_Y}(A, \pi_* \text{Hom}(V, F)) \cong \text{Hom}_{\Lambda}(A \otimes \Lambda, t_V(F))$$
$$\cong \text{Hom}_{\Lambda}(t'^{-1}_V(A \otimes \Lambda), t'^{-1}_V t_V(F))$$
$$\cong \text{Hom}_{\Lambda}(\pi^* A \otimes V, t'^{-1}_V t_V(F))$$

where we use the projection formula, the adjunction $\Lambda \otimes \mathcal{O}_Y \dashv \text{For}$, the fact that $t'^{-1}_V$ is an equivalence, hence fully faithful, and the $Y$-linearity of $t'^{-1}_V$.

Since the right-admissibile $Y$-linear subcategory $\mathcal{V} \subset \mathcal{D}^b(X)$ contains the relative spanning class $V$, we get $\mathcal{V} = \mathcal{D}^b(X)$ by Lemma 2.2.5. This shows that $V$ is a relative generator and that $t_V = t'_V$ is an equivalence. \qed

Let $\pi' : X' \to Y$ be a second proper morphism and let $\Phi : \mathcal{D}^b(X) \xrightarrow{\sim} \mathcal{D}^b(X')$ be a relative FM transform.

Lemma 2.2.9. If

$$\bar{\Phi}_\Lambda := \bar{\Phi}(V, V) : \Lambda_V = \pi_* \text{Hom}(V, V) \to \pi'_* \text{Hom}(\Phi(V), \Phi(V)) = \Lambda_{\Phi(V)}$$

is an isomorphism, then the following diagram of functors commutes:

$$\begin{array}{ccc}
\mathcal{D}^b(X) & \xrightarrow{t_V} & \mathcal{D}^b(\Lambda_V) \\
\Phi \downarrow & & \downarrow \otimes_{\Lambda_V} \Lambda_{\Phi(V)} \\
\mathcal{D}^b(X') & \xrightarrow{t_{\Phi(V)}} & \mathcal{D}^b(\Lambda_{\Phi(V)})
\end{array}$$

(2.6)
Proof. We first show that $\Phi(V, E) : t_V(E) \to t_{\Phi(V)}(\Phi(E))$ is an isomorphism in $D^b(Y)$ for every $E \in D(X)$. Assume first that there is an exact triangle $\pi^* A \otimes V \to E \to \pi^* B \otimes V$ for some $A, B \in D^{\text{perf}}(Y)$ and consider the induced morphism of triangles

\[
\begin{array}{ccc}
\pi_* \text{Hom}(V, \pi^* A \otimes V) & \xrightarrow{\Phi(V, \pi^* A \otimes V)} & \pi_* \text{Hom}(V, E) \\
\downarrow & & \downarrow \\
\pi'_* \text{Hom}(\Phi(V), \Phi(\pi^* A \otimes V)) & \xrightarrow{\Phi(V, \pi^* A \otimes V)} & \pi'_* \text{Hom}(\Phi(V), \Phi(E)) \\
\end{array}
\]

The outer vertical arrows are isomorphisms because they decompose as

\[
\pi_* \text{Hom}(V, V \otimes \pi^* A) \xrightarrow{\Phi(V, \pi^* A \otimes V)} \pi_* \text{Hom}(V, V) \otimes A \xrightarrow{\Phi_A} \pi'_* \text{Hom}(\Phi(V), \Phi(V)) \otimes A
\]

Therefore, the middle vertical arrow is an isomorphism as well. Since $V$ is a relative generator, we can show that $\Phi(V, E)$ is an isomorphism for arbitrary $E \in D^b(X)$ by repeating the above argument.

Using the commutativity of (2.5) with $E = F = G = V$, we see that $\Phi(V, E)$ induces an $\Lambda_{\Phi(V)}$-linear isomorphism $\pi_* \text{Hom}(V, E) \otimes_{\Lambda_V} \Lambda_{\Phi(V)} \xrightarrow{\Phi} \pi'_* \text{Hom}(\Phi(V), \Phi(E))$. □

Lemma 2.2.10. The functor $\Phi$ is fully faithful if and only if $\Phi : \Lambda_V \to \Lambda_{\Phi(V)}$ is an isomorphism.

Proof. If $\Phi$ is fully faithful, the unit $\text{id} \to \Phi^R \Phi$ is an isomorphism. Hence, $\Phi$ is an isomorphism; see (2.4).

Conversely, let $\Phi$ be an equivalence. By Lemma 2.2.9, we get a commutative diagram

\[
\begin{array}{ccc}
D^b(X) & \xrightarrow{t_V} & D^b(\Lambda_V) \\
\downarrow \Phi & & \downarrow \otimes_{\Lambda_V} \Lambda_{\Phi(V)} \\
\langle \Phi(V) \rangle & \xrightarrow{t_{\Phi(V)}} & D^b(\Lambda_{\Phi(V)})
\end{array}
\]

In this diagram, the horizontal functors are tilting equivalences. The right-hand vertical functor is an equivalence, too, by assumption on $\Phi$. Hence, $\Phi : D^b(X) \to \langle \Phi(V) \rangle$ is an equivalence, which implies that $\Phi : D^b(X) \to D^b(X')$ is fully faithful. □

Lemma 2.2.11. Let $V \in D^b(X)$ be a relative tilting object, $\Phi : D^b(X) \xrightarrow{\sim} D^b(X)$ a relative FM autoequivalence, and $\nu : V \xrightarrow{\sim} \Phi(V)$ an isomorphism such that $\Phi_{\nu} = \nu \circ \nu^{-1} : \pi_* \text{Hom}(V, V) \to \pi'_* \text{Hom}(\Phi(V), \Phi(V))$, i.e. $\Phi_{\nu}(\varphi) \circ \nu = \nu \circ \varphi$ for all open subsets $U \subset Y$ and $\varphi \in \Lambda_V(U)$. Then there exists an isomorphism of functors $\text{id} \xrightarrow{\sim} \Phi$ restricting to $\nu$.

Proof. We claim that, under our assumptions, the following diagram of functors commutes

\[
\begin{array}{ccc}
D^b(X) & \xrightarrow{t_V} & D^b(\Lambda_V) \\
\downarrow \text{id} & & \downarrow \otimes_{\Lambda_V} \Lambda_{\Phi(V)} \\
D^b(X) & \xrightarrow{t_{\Phi(V)}} & D^b(\Lambda_{\Phi(V)})
\end{array}
\]
We construct a natural isomorphism $\eta: \theta \circ \eta \otimes \lambda \Lambda_{\Phi}^V$ as follows. For $E \in D^b(X)$, there is a natural $O_Y$-linear isomorphism $\pi_* \mathcal{H}(\Phi(V), E) \rightarrow \pi_* \mathcal{H}(\Phi(V), E) \otimes \Lambda_{\Phi}^V$ given by $f \mapsto fu \otimes 1$; the inverse map is $g \otimes 1 \mapsto gu^{-1}$. This map is linear over $\Lambda_{\Phi}^V$ because, for a local section $\lambda \in \pi_* \mathcal{H}(\Phi(V), \Phi(V))$, we have by our assumption, setting $\varphi = \Phi^{-1}(\lambda)$:

$$\eta(f \lambda) = f \lambda \otimes 1 = fu \Phi^{-1}(\lambda) \otimes 1 \Rightarrow f \nu \otimes \lambda = (f \nu \otimes 1) \lambda.$$ 

Comparing the diagrams (2.7) and (2.6) shows that $\Phi \cong \text{id}$.

**Corollary 2.2.12.** Let $V \in D^b(X)$ be a relative tilting object, $\Phi_1, \Phi_2: D^b(X) \rightarrow D^b(Y)$ relative FM equivalences, and $\nu: \Phi_1(V) \rightarrow \Phi_2(V)$ an isomorphism such that $\Phi_2\nu(\varphi) \circ \nu = \nu \circ \Phi_1 \nu(\varphi)$ for all $\varphi \in \Lambda_Y(U)$ and $U \subset Y$ open. Then there exists a isomorphism of functors $\Phi_1 \cong \Phi_2$ restricting to $\nu$.

Moreover, if $V = L_1 \oplus \cdots \oplus L_k$ decomposes as a direct sum, then the above condition is satisfied by specifying isomorphisms $\nu_i: \Phi_1(L_i) \cong \Phi_2(L_i)$ inducing functor isomorphisms $\Phi_1 \nu \circ \nu = \nu \circ \Phi_1 \nu(\varphi)$ for all $\varphi \in \Lambda_Y(U)$ and $U \subset Y$ open. Then there exists a isomorphism of functors $\Phi_1 \cong \Phi_2$ restricting to $\nu$.

**Remark 2.2.13.** All the results of this subsection remain valid in an equivariant setting, where a finite group $G$ acts on $X$ and $\pi: X \rightarrow Y$ is $G$-invariant. Then the correct sheaf of $O_Y$-algebras is $\Lambda_Y = \pi_*^G \mathcal{H}(V, V)$.

### 2.2.8 Spherical functors

An exact functor $\varphi: \mathcal{C} \rightarrow \mathcal{D}$ between triangulated categories is called spherical if it admits both adjoints, if the cone endofunctor $F[1] := \text{cone}(\text{id}_C \rightarrow \varphi \varphi)$ is an autoequivalence of $\mathcal{C}$, and if the canonical functor morphism $\varphi \varphi \rightarrow F[1]$ is an isomorphism. A spherical functor is called split if the triangle defining $F$ is split. The proper framework for dealing with functorial cones are dg-categories; the triangulated categories in this article are of geometric nature, and we can use Fourier–Mukai transforms. See [AL13] for proofs in general generality.

Given a spherical functor $\varphi: \mathcal{C} \rightarrow \mathcal{D}$, the cone of the natural transformation $T = T_{\varphi} := \text{cone}(\varphi \varphi \rightarrow \text{id}_D)$ is called the twist around $\varphi$; it is an autoequivalence of $\mathcal{D}$.

The following lemma follows immediately from the definition, since an equivalence has its inverse functor as both left and right adjoint.

**Lemma 2.2.14.** Let $\varphi: \mathcal{C} \rightarrow \mathcal{D}$ be a spherical functor and let $\delta: \mathcal{D} \rightarrow \mathcal{D}'$ be an equivalence. Then $\delta \circ \varphi: \mathcal{C} \rightarrow \mathcal{D}'$ is a spherical functor with associated twist functor $T_{\delta \varphi} = \delta T_{\varphi} \delta^{-1}$.

### 2.3 The geometric setup

Let $X$ be a smooth quasi-projective variety together with an action of a finite group $G$. Let $S := \text{Fix}(G)$ be the locus of fixed points. Then $S \subset X$ is a closed subset, which is automatically smooth since, locally in the analytic topology, the action can be linearised by Cartan’s lemma, see [Car57, Lem. 2]. Also note that $X/G$ has rational singularities, like any quotient singularity over $\mathbb{C}$ [Kov00].

**Condition 2.3.1.** We make strong assumptions on the group action:

1. $G \cong \mu_m$ is a cyclic group. Fix a generator $g \in G$.

2. Only the trivial isotropic groups $1$ and $\mu_m$ occur.
3. The generator $g$ acts on the normal bundle $N := N_{S/X}$ by multiplication with some
fixed primitive $m$-th root of unity $\zeta$.

Condition (ii) obviously holds if $m$ is prime.

Condition (iii) can be rephrased: there is a splitting $T_{X|S} = T_S \oplus N_{S/X}$ because $T_S$ is
the subsheaf of $G$-invariants of $T_{X|S}$ and we work over characteristic 0. By (iii), this is even
the splitting into the eigenbundles corresponding to the eigenvalues 1 and $\zeta$. We denote by
$\chi: G \to \mathbb{C}^*$ the character with $\chi(g) = \zeta^{-1}$. Hence, we can reformulate (iii) by saying that $G$
acts on $N$ via $\chi^{-1}$.

From these assumptions we deduce the following commutative diagram

$$
\begin{array}{ccc}
\bar{X} & \xrightarrow{p} & X \\
\downarrow j & & \downarrow a \\
\bar{X}/G = \bar{Y} & \xrightarrow{i\text{ blow-up in } S} & S \\
\downarrow q & & \downarrow b \\
Z = \mathbb{P}(N) & \xrightarrow{\nu} & Y = X/G \\
\end{array}
$$

(2.8)

where $a$, $b$, $i$, and $j$ are closed embeddings and $\pi$ is the quotient morphism. The $G$-action on
$X$ lifts to a $G$-action on $\bar{X}$. Since, by assumption, $G$ acts diagonally on $N$, it acts trivially on
the exceptional divisor $Z = \mathbb{P}(N)$. In particular, the fixed point locus of the $G$-action on $\bar{X}$ is
a divisor. Hence, the quotient variety $\bar{Y}$ is again smooth and the quotient morphism $q$ is flat
due to the Chevalley–Shephard–Todd theorem. Since the composition $\pi \circ p$ is $G$-invariant,
it induces the morphism $\varrho: \bar{Y} \to Y$ which is easily seen to be birational, hence a resolution
of singularities. The preimage $\varrho^{-1}(S)$ of the singular locus is a divisor in $\bar{Y}$. Hence, by the
universal property of the blow-up, we get a morphism $\bar{Y} \to \text{Bl}_S Y$ which is easily seen to be
an isomorphism.

2.3.1 The resolution as a moduli space of $G$-clusters

The result of this section might be of independent interest. Let $X$ be a smooth quasi-projective
variety and $G$ a finite group acting on $X$. A $G$-cluster on $X$ is a closed zero-dimensional
$G$-invariant subscheme $W \subset X$ such that the $G$-representation $H^0(W, \mathcal{O}_W)$ is isomorphic
to the regular representation of $G$. There is a fine moduli space $\text{Hilb}^G(X)$ of $G$-clusters,
called the $G$-Hilbert scheme. It is equipped with the equivariant Hilbert–Chow morphism
$\tau: \text{Hilb}^G(X) \to X/G, W \mapsto \text{supp}(W)$, mapping $G$-clusters to their underlying $G$-orbits.

**Proposition 2.3.2.** Let $G$ be a finite cyclic group acting on $X$ such that all isotropy groups
are either 1 or $G$, and such that $G$ acts on the normal bundle $N_{\text{Fix}(G)/X}$ by scalars which means
that Condition 2.3.1 is satisfied. Then there is an isomorphism

$$\varphi: \bar{Y} \xrightarrow{\cong} \text{Hilb}^G(X) \quad \text{with} \quad \tau \circ \varphi = \varrho.$$ 

**Proof.** We use the notation from (2.8). One can identify $\bar{X}$ with the reduced fibre product
$(\bar{Y} \times_Y X)_{\text{red}}$ which gives a canonical embedding $\bar{X} \subset \bar{Y} \times X$. Under this embedding, the generic
fibre of $q$ is a reduced free $G$-orbit of the action on $X$. In particular, it is a $G$-cluster. By the
flatness of $q$, every fibre is a $G$-cluster and we get the classifying morphism $\varphi: \bar{Y} \to \text{Hilb}^G(X)$
which is easily seen to satisfy $\tau \circ \varphi = \varrho$. 75
Let \( s \in S \) and \( z \in Z \) with \( \nu(z) \in s \). Let \( \ell \subset N(s) \) be the line corresponding to \( z \). Then, one can check that the tangent space of the \( G \)-cluster \( q^{-1}(i(z)) \subset X \) is exactly \( \ell \). Hence, the \( G \)-clusters in the family \( \tilde{X} \) are all different so that the classifying morphism \( \varphi \) is injective. For the bijectivity of \( \varphi \), it is only left to show that the \( G \)-orbits supported on a given fixed point \( s \in S \) are parametrised by \( \mathbb{P}(N(s)) \). Let \( \xi \subset X \) be such a \( G \)-cluster. In particular, \( \xi \) is a length \( m = |G| \) subscheme concentrated in \( s \) and hence can be identified with an ideal \( I \subset \mathcal{O}_{X,s}/m^{\infty}_{X,s} \) of codimension \( m \). By Cartan’s lemma, the \( G \)-action on \( X \) can be linearised in an analytic neighbourhood of \( s \). Hence, there is an \( G \)-equivariant isomorphism

\[
\mathcal{O}_{X,s}/m^{\infty}_{X,s} \cong \mathbb{C}[x_1, \ldots, x_k, y_1, \ldots, y_n]/(x_1, \ldots, x_k, y_1, \ldots, y_n)^m =: R
\]

where \( G \) acts trivially on the \( x_i \) and by multiplication by \( \zeta^{-1} \) on the \( y_i \). Furthermore, \( n = \text{rank} N_{S/X} \) and \( k = \text{rank} T_S = \dim X - n \). By assumption, \( \mathcal{O}(\xi) \) is the regular \( \mu_m \)-representation. In other words,

\[
\mathcal{O}(\xi) \cong R/I \cong \chi^0 \oplus \chi \oplus \cdots \oplus \chi^{m-1} \tag{2.9}
\]

where \( \chi \) is the character given by multiplication by \( \zeta^{-1} \). In particular, \( R/I \) has a one-dimensional subspace of invariants. It follows that every \( x_i \) is congruent to a constant polynomial modulo \( I \). Hence, we can make an identification \( \mathcal{O}(\xi) \cong R'/J \) where \( J \) is a \( G \)-invariant ideal in \( R' = \mathbb{C}[y_1, \ldots, y_k]/n^m \) where \( n = (y_1, \ldots, y_n) \). The decomposition of the \( G \)-representation \( R' \) into eigenspaces is exactly the decomposition into the spaces of homogeneous polynomials. Hence, an ideal \( J \subset R' \) is \( G \)-invariant if and only if it is homogeneous. Furthermore, (2.9) implies that

\[
\dim_{\mathbb{C}}(n^i/(J \cap n^i + n^{i+1})) = 1 \quad \text{for all } i = 0, \ldots, m - 1
\]

which means that \( \xi \) is curvilinear. In summary \( \xi \) can be identified with a homogeneous curvilinear ideal \( J \) in \( R' \). The choice of such a \( J \) corresponds to a point in \( \mathbb{P}((n/n^2)^\vee) \cong \mathbb{P}(N(s)) \); see [Göt 94, Rem. 2.1.7].

Hence, \( \varphi \) is a bijection and we only need to show that \( \text{Hilb}^G(X) \) is smooth. The smoothness in points representing free orbits is clear since the \( G \)-Hilbert–Chow morphism is an isomorphism on the locus of these points. So it is sufficient to show that

\[
\text{Hom}_{\mathcal{O}_{G,\nu(X)}}^1(\mathcal{O}_\xi, \mathcal{O}_\xi) = \dim X = n + k
\]

for a \( G \)-cluster \( \xi \) supported on a fixed point. Following the above arguments, we have

\[
\text{Hom}_{\mathcal{O}_{G,\nu(X)}}^* (\mathcal{O}_\xi, \mathcal{O}_\xi) \cong \text{Hom}^*_{\mathcal{O}_{\mathbb{A}^k \times \mathbb{A}^n}}(\mathcal{O}_{\xi'}, \mathcal{O}_{\xi'})
\]

where \( G \) acts trivially on \( \mathbb{A}^k \) and by multiplication by \( \zeta \) on \( \mathbb{A}^n \). Furthermore, by a transformation of coordinates, we may assume that

\[
\xi' = V(x_1, \ldots, x_k, y_1^m, y_2, \ldots, y_n) \subset \mathbb{A}^k \times \mathbb{A}^n.
\]

We have \( \mathcal{O}_\xi' \cong \mathcal{O}_0 \boxtimes \mathcal{O}_\eta \) where

\[
\eta = V(y_1^m, y_2, \ldots, y_n) \subset \mathbb{A}^n.
\]
By Künneth formula, we get
\[
\text{Hom}^*_{D^{b}(\mathbb{A}^k \times \mathbb{A}^n)}(O_{\ell'}, O_{\ell'}) \cong \text{Hom}^*_{D^{b}(\mathbb{A}^k)}(O_0, O_0) \otimes \text{Hom}^*_{D^{b}(\mathbb{A}^n)}(O_\eta, O_\eta) \\
\cong \wedge^*(C^k) \otimes \text{Hom}^*_{D^{b}(\mathbb{A}^n)}(O_\eta, O_\eta).
\]
Furthermore, \(\text{Hom}^0_{D^{b}(\mathbb{A}^n)}(O_\eta, O_\eta) \cong H^0(O_\eta)^G \cong \mathbb{C}.\) Hence, it is sufficient to show that \(\text{Hom}^1_{D^{b}(\mathbb{A}^n)}(O_\eta, O_\eta) \cong \mathbb{C}^m.\) Note that \(\eta\) is contained in the line \(\ell = V(y_2, \ldots, y_n).\) On \(\ell\) we have the Koszul resolution
\[
0 \to O_\ell \xrightarrow{y_1^m} O_\ell \to O_\eta \to 0.
\]
Using this, we compute
\[
\text{Hom}^*_{D^{b}(\ell)}(O_\eta, O_\eta) \cong O_\eta[0] \oplus O_\eta[-1].
\]
Note that the normal bundle of \(\ell,\) as an equivariant bundle, is given by \(N_{\ell/\mathbb{A}^n} \cong (O_{\ell} \otimes \chi^{-1})^\oplus n-1.\) By [AC12, Thm. 1.4], we have
\[
\text{Hom}^*_{D^{b}(\mathbb{A}^n)}(O_\eta, O_\eta) \cong \text{Hom}^*_{D^{b}(\ell)}(O_\eta, O_\eta \otimes \wedge^* N_{\ell/\mathbb{A}^n}) \\
\cong \text{Hom}^*_{D^{b}(\ell)}(O_\eta, O_\eta) \otimes \wedge^*((O_{\ell} \otimes \chi^{-1})^\oplus n-1).
\]
Evaluating in degree 1 gives
\[
\text{Hom}^1_{D^{b}(\mathbb{A}^n)}(O_\eta, O_\eta) \cong O_\eta \oplus (O_\eta \otimes \chi^{-1})^\oplus n-1.
\]
Since, as a \(G\)-representation, \(O_\eta \cong \chi^0 \oplus \chi^1 \oplus \cdots \oplus \chi^{m-1},\) we get an \(n\)-dimensional space of invariants
\[
\text{Hom}^1_{D^{b}(\mathbb{A}^n)}(O_\eta, O_\eta) \cong \text{Hom}^1_{D^{b}(\mathbb{A}^n)}(O_\eta, O_\eta)^G \cong \mathbb{C}^n.
\]

The following lemma is needed later in Subsection 2.4.4 but its proof fits better into this section.

**Lemma 2.3.3.** Assume that \(m = |G| \geq n = \text{codim}(S \hookrightarrow X).\) Let \(\xi_1, \xi_2 \subset X\) be two different \(G\)-clusters supported on the same point \(s \in S.\) Then \(\text{Hom}^*_{D^{b}(\mathbb{A}^n)}(O_{\xi_1}, O_{\xi_2}) = 0.\)

**Proof.** By the same arguments as in the proof of the previous proposition we can reduce to the claim that
\[
\text{Hom}^*_{D^{b}(\mathbb{A}^n)}(O_{\eta_1}, O_{\eta_2}) = 0
\]
where \(\eta_1 = V(y_1^n, y_2, \ldots, y_n)\) and \(\eta_2 = V(y_1, y_2^n, y_3, \ldots, y_n).\) Set \(\ell_1 = V(y_2, \ldots, y_n), \ell_2 = V(y_1, y_3, \ldots, y_n), E = (\ell_1, \ell_2) = V(y_3, \ldots, y_n)\) and consider the diagram of closed embeddings
\[
\begin{array}{ccc}
\{0\} & \xrightarrow{\ell_2} & E \\
\downarrow{\eta_1} & & \downarrow{E} \\
\ell_1 & \xrightarrow{\ell_1} & \mathbb{A}^n
\end{array}
\]
\[
\begin{array}{ccc}
\{0\} & \xrightarrow{\ell_2} & E \\
\downarrow{\eta_2} & & \downarrow{E} \\
\ell_1 & \xrightarrow{\ell_2} & \mathbb{A}^n
\end{array}
\]
where \( N_t \cong (O_E \otimes \chi^{-1})^{\oplus n-2} \). By [7, Lem. 3.3] (alternatively, one may consult [Gri13] or [ACH14] for more general results on derived intersection theory), we get

\[
\text{Hom}^*_\mathbb{D}(\mathbb{A}^n)(O_{\eta_1}, O_{\eta_2}) \cong \text{Hom}^*_{\mathbb{D}}(t_1^* O_{\eta_1}, t_2^* O_{\eta_2}) \\
\cong \text{Hom}^*_{\mathbb{D}}(t_2^* t_1^* O_{\eta_1}, O_{\eta_2}) \\
\cong \text{Hom}^*_{\mathbb{D}}(u^* v^* O_{\eta_1}, O_{\eta_2}) \otimes \wedge^s N_{t_2} t_1 \\
\cong \text{Hom}^*_{\mathbb{D}}(O_0, O_{\eta_2}) \otimes \wedge^s (O_{t_2} \otimes \chi^{-1})^{\oplus n-2}. \tag{2.10}
\]

We have the equivariant Koszul resolution

\[ 0 \to O_{\ell} \otimes \chi \xrightarrow{y} O_{\ell} \to 0 \]

of \( O_0 \) where we set for simplicity \( \ell := \ell_2 \) and \( y := y_2 \). Applying Hom(\( \_ \), \( O_{\eta} \)) we get

\[ 0 \to \mathbb{C} [y]/y^m \otimes \xrightarrow{y} \mathbb{C} [y]/y^m \otimes \chi^{-1} \to 0. \]

Taking cohomology, we get

\[ \text{Hom}^*_{\mathbb{D}}(O_0, O_{\eta_2}) \cong \mathbb{C} [y^{m-1}]/[0] \oplus \mathbb{C} (1) \otimes \chi^{-1} [-1] \cong O_0 \otimes \chi^{-1} [0] \oplus O_0 \otimes \chi^{-1} [-1]. \]

It follows that the graded vector space (2.10) does not have invariants (recall that \( m \geq n \)). \( \square \)

### 2.4 Proof of the main result

In this section, we will study the derived categories \( \mathbb{D}(\tilde{Y}) \) and \( \mathbb{D}_G(X) \) in the setup described in the previous section. In particular, we will prove Theorems 2.1.2 and 2.1.3.

We set \( n = \text{codim}(S \hookrightarrow X) \) and \( m = |G| \), in other words \( G = \mu_m \). We consider, for \( \alpha \in \mathbb{Z}/m\mathbb{Z} \) and \( \beta \in \mathbb{Z} \), the exact functors

\[
\Phi := p_* \circ q^* \circ \text{triv}: \mathbb{D}(\tilde{Y}) \to \mathbb{D}_G(X) \\
\Psi := (-)^G \circ q_* \circ p^*: \mathbb{D}_G(X) \to \mathbb{D}(\tilde{Y}) \\
\Theta_\beta := i_* (v^* (\_ \otimes O_\nu(\beta))): \mathbb{D}(S) \to \mathbb{D}(\tilde{Y}) \\
\Xi_\alpha := (a_* \circ \text{triv}) \otimes \chi^\alpha: \mathbb{D}(S) \to \mathbb{D}_G(X).
\]

With this notation, the precise version of Theorem 2.1.2 is

**Theorem 2.4.1.**

1. The functor \( \Phi \) is fully faithful for \( m \geq n \) and an equivalence for \( m = n \). For \( m > n \), all the functors \( \Xi_\alpha \) are fully faithful and there is a semi-orthogonal decomposition

\[ \mathbb{D}_G(X) = \langle \Xi_{-m}(\mathbb{D}(S)), \ldots, \Xi_1(\mathbb{D}(S)), \Phi(\mathbb{D}(\tilde{Y})) \rangle. \]

2. The functor \( \Psi \) is fully faithful for \( n \geq m \) and an equivalence for \( n = m \). For \( n > m \), all the functors \( \Theta_\beta \) are fully faithful and there is a semi-orthogonal decomposition

\[ \mathbb{D}(\tilde{Y}) = \langle \Theta_{m-n}(\mathbb{D}(S)), \ldots, \Theta_{-1}(\mathbb{D}(S)), \Psi(\mathbb{D}_G(X)) \rangle. \]

**Remark 2.4.2.** We will see later in Corollary 2.4.14 that \( K_{\tilde{Y}} \leq g^* K_Y \) for \( m \geq n \) and \( K_{\tilde{Y}} \geq g^* K_Y \) for \( n \geq m \). Hence, Theorem 2.4.1 is in accordance with the DK-Hypothesis as described in the introduction.

For the proof, we first need some more preparations.
2.4.1 Generators and linearity

**Lemma 2.4.3.** The bundle \( V := O_X \otimes \mathbb{C}[G] = O_X \otimes (\chi^0 \oplus \cdots \oplus \chi^{m-1}) \) is a relative tilting object for \( D^b_G(X) \) over \( D^\text{perf}(Y) \).

*Proof.* If \( L \in \text{Pic}(Y) \subset D^\text{perf}(Y) \) is an ample line bundle, then so is \( \pi^*(L) \). Hence, \( D^b(X) \) has a generator of the form \( E := \pi^*(O_Y \oplus L \oplus \cdots \oplus L^\otimes k) \) for some \( k \gg 0 \); see [Orl09].

In particular, \( E \) is a spanning class of \( D^b(Y) \). Using the adjunction \( \text{Res} \dashv \text{Ind} \), it follows that \( \text{Ind}(E) \cong E \oplus E \otimes \chi \oplus \cdots \oplus E \otimes \chi^{m-1} \) is a spanning class of \( D^b_G(X) \). Hence, \( V = \text{Ind}(O_X) \) is a relative spanning class of \( D^b_G(X) \) over \( D^\text{perf}(Y) \).

Since \( V \) is a vector bundle, so is \( \text{Hom}(V,V) = V^\vee \otimes V \). The map \( \pi \) is finite, hence \( \pi_* \) is exact (does not need to be derived). Finally, taking \( G \)-invariants is exact because we work in characteristic 0. Altogether, \( \pi_* \text{Hom}(V,V) \) is a sheaf concentrated in degree 0. \( \Box \)

**Lemma 2.4.4.** The functors \( \Phi \) and \( \Psi \), and for all \( \alpha, \beta \in \mathbb{Z} \) the subcategories

\[
\Xi_\alpha(D^b(S)) = a_*(D^b(S)) \otimes \chi^\alpha \subset D^b_G(X)
\]

\[
\Theta_\beta(D^b(S)) = i_* \nu^* D^b(S) \otimes O_{\tilde{Y}}(\beta) \subset D^b(\tilde{Y})
\]

are \( Y \)-linear for \( \pi^* \text{triv} : D^\text{perf}(Y) \to D^b_G(X) \) and \( q^* : D^\text{perf}(Y) \to D^b(\tilde{Y}) \), respectively.

*Proof.* We first show that \( \Phi \) is \( Y \)-linear. Recall that in our setup this means

\[
\Phi(q^*(E) \otimes F) \cong \pi^* \text{triv}(E) \otimes \Phi(F)
\]

for any \( E \in D^\text{perf}(Y) \) and \( F \in D^b(\tilde{Y}) \). But this holds, since

\[
\pi^* \text{triv}(E) \otimes \Phi(F) \cong \pi^* \text{triv}(E) \otimes p_* q^* \text{triv}(F)
\]

\[
\cong p_*(p^* \pi^* \text{triv}(E) \otimes q^* \text{triv}(F))
\]

\[
\cong p_*(q^* \pi^* \text{triv}(E) \otimes q^* \text{triv}(F))
\]

\[
\cong p_* q^* \text{triv}(q^*(E) \otimes F).
\]

The proof that \( \Psi \) is \( Y \)-linear is similar and is left to the reader.

The \( Y \)-linearity of the image categories follows from Lemma 2.2.5(i). \( \Box \)

**Lemma 2.4.5.** The set of sheaves \( S := \{ O_{\bar{Y}} \} \cup \{ i_s \Omega^r \mid s \in S, r = 0, \ldots, n-1 \} \) forms a spanning class of \( D^b(\tilde{Y}) \) over \( Y \), where \( i_s : \mathbb{P}^{n-1} \cong q^{-1}(s) \hookrightarrow \tilde{Y} \) denotes the fibre embedding.

*Proof.* We need to show that \( \hat{S} := q^* D^\text{perf}(Y) \otimes S \) is a spanning class of \( D^b(\tilde{Y}) \). Let \( \tilde{y} \in \tilde{Y} \setminus Z \). Then \( y = q(\tilde{y}) \) is a smooth point of \( Y \). Hence, \( O_{\tilde{Y}} \in D^\text{perf}(Y) \) and \( O_{\tilde{y}} \in q^* D^\text{perf}(Y) = q^* D^\text{perf}(Y) \otimes O_{\bar{Y}} \subset \hat{S} \). Thus, an object \( E \in D^b(\tilde{Y}) \) with \( \text{supp} E \cap (\tilde{Y} \setminus Z) \neq \emptyset \) satisfies \( \text{Hom}^*(E, \hat{S}) \neq 0 \neq \text{Hom}^*(\hat{S}, E) \); see [Huy06, Lemma 3.29].

Let now \( 0 \neq E \in D^b(\tilde{Y}) \) with \( \text{supp} E \subset Z \). Then there exists \( s \in S \) such that \( i_s^* E \neq 0 \neq i^*_s E \); see again [Huy06, Lemma 3.29]. Since the \( \Omega^r(\tilde{Y}) \) form a spanning class of \( \mathbb{P}^{n-1} \), we get by adjunction \( \text{Hom}^*(E, S) \neq 0 \neq \text{Hom}^*(S, E) \). \( \Box \)
2.4.2 On the equivariant blow-up

Recall that the blow-up morphism $q: \widetilde{X} \to X$ is $G$-equivariant. Let $L_{\widetilde{X}} \in \text{Pic}^G(\widetilde{X})$ (we will sometimes simply write $L$ instead of $L_{\widetilde{X}}$) be the equivariant line bundle $O_{\widetilde{X}}(Z)$ equipped with the unique linearisation whose restriction to $Z$ gives the trivial action on $O_Z(Z) \cong O_\nu(-1)$. We consider a point $z \in Z$ with $\nu(z) = s$ corresponding to a line $\ell \subset N_{Z/X}(s)$. Then the normal space $N_{Z/\widetilde{X}}(z)$ can be equivariantly identified with $\ell$. It follows by Condition 2.3.1 that $N_{Z/\widetilde{X}} \cong (L_{\widetilde{X}} \otimes \chi^{-1})|_Z$ as an equivariant bundle. Hence, in $\text{Coh}^G(\widetilde{X})$, there is the exact sequence

$$0 \to L_{\widetilde{X}}^{-1} \otimes \chi \to O_{\widetilde{X}} \to O_Z \to 0 \quad (2.11)$$

where both $O_{\widetilde{X}}$ and $O_Z$ are equipped with the canonical linearisation, which is the one given by the trivial action over $Z$.

Lemma 2.4.6. For $\ell = 0, \ldots, n - 1$ we have $p_*L_{\widetilde{X}}^\ell = O_X \otimes \chi^\ell$.

Proof. We have $p_*O_{\widetilde{X}} \cong O_X$, both, $O_{\widetilde{X}}$ and $O_X$, equipped with the canonical linearisations. Hence, the assertion is true for $\ell = 0$. By induction, we may assume that $p_*L_{\widetilde{X}}^\ell \cong O_X \otimes \chi^\ell$.

We tensor (2.11) by $L_{\widetilde{X}}^\ell$ to get

$$0 \to L_{\widetilde{X}}^{\ell-1} \otimes \chi \to L_{\widetilde{X}}^\ell \to O_\nu(-\ell) \to 0.$$ 

Since $0 \leq \ell \leq n - 1$, we have $p_*O_\nu(-\ell) = 0$. Hence, we get an isomorphism

$$p_*(L_{\widetilde{X}}^\ell) \cong p_*(L_{\widetilde{X}}^{\ell-1} \otimes \chi) \cong p_*(L_{\widetilde{X}}^{\ell-1}) \otimes \chi \cong O_X \otimes \chi^{\ell-1} \otimes \chi \cong O_X \otimes \chi^\ell.$$ 

Lemma 2.4.7. The smooth blow-up $p: \widetilde{X} \to X$ has $G$-linearised relative dualising sheaf

$$\omega_p \cong L_{\widetilde{X}}^{n-1} \otimes \chi^{1-n} = L_{\widetilde{X}}^{n-1} \otimes \chi \in \text{Pic}^G(\widetilde{X}).$$

Proof. The non-equivariant relative dualising sheaf of the blow-up is $\omega_p \cong O_{\widetilde{X}}((n-1)Z)$. Since $p$ is $G$-equivariant, there is a unique linearisation of $\omega_p$ such that $p^! = p^* \otimes p_*: D^b_G(X) \to D^b_G(\widetilde{X})$ is the right-adjoint of $p_*: D^b_G(\widetilde{X}) \to D^b_G(X)$. In the following, we will compute this linearisation of $\omega_p$.

As the equivariant pull-back $p^!$ is fully faithful, $p^!: D^b_G(X) \to D^b_G(\widetilde{X})$ is fully faithful, too. Hence, adjunction gives an isomorphism of equivariant sheaves, $p_*\omega_p \cong p_*p^!O_X \cong O_X$. The claim now follows from Lemma 2.4.6.

We denote by $i_s: \mathbb{P}^{n-1} \cong g^{-1}(s) \hookrightarrow \tilde{Y}$ the embedding of the fibre of $g$ and by $j_s: \mathbb{P}^{n-1} \cong p^{-1}(s) \hookrightarrow \widetilde{X}$ the embedding of the fibre of $p$ over $s \in S$.

Lemma 2.4.8. Let $s \in S$ and $r = 0, \ldots, n - 1$,

$$\mathcal{H}^{-r}(p^*O_s) \cong j_{ss}(\Omega^r(r) \otimes \chi^r).$$

Proof. It is well known that, for the underlying non-equivariant sheaves, we have $\mathcal{H}^{-r}(p^*O_s) \cong j_{ss}(\Omega^r(r))$; see [Huy06, Prop. 11.12]. Since the sheaves $\Omega^r(r)$ are simple, i.e. $\text{End}(\Omega^r(r)) = \mathbb{C}$, we have $\mathcal{H}^{-r}(p^*O_s) \cong j_{ss}(\Omega^r(r) \otimes \chi^\alpha_r)$ for some $\alpha_r \in \mathbb{Z}/m\mathbb{Z}$. So we only need to show $\alpha_r = r$. 

80
Let \( r \in \{0, \ldots, n-1\} \). We have \( p_r \mathcal{L}^{-r} \cong p_*(\mathcal{L}^{-r+n-1} \otimes \chi^{1-n}) \) by Lemma 2.4.7. Since \( -r + n - 1 \in \{0, \ldots, n-1\} \), Lemma 2.4.6 gives \( p_r \mathcal{L}^{-r} \cong \mathcal{O}_X \otimes \chi^{-r} \). By adjunction,

\[
\mathbb{C}[0] \cong \text{Hom}^*_{\mathbb{D}^b_G(X)}(\mathcal{O}_X \otimes \chi^{-r}, \mathcal{O}_s \otimes \chi^{-r}) \cong \text{Hom}^*_{\mathbb{D}^b_G(\tilde{X})}(\mathcal{L}^{-r}, p^* \mathcal{O}_s \otimes \chi^{-r}) .
\]

By Lemma 2.2.3, for \( r \neq v \), we have

\[
\text{Hom}^*_{\mathbb{D}^b(\tilde{X})}(\mathcal{O}_{\tilde{X}}(-rZ), j_* s_* \Omega^r(v)) \cong \text{Hom}^*_{\mathbb{D}^b(\mathbb{P}^{n-1})}(\mathcal{O}(r), \Omega^r(v)) = 0 .
\]

Using the spectral sequence in \( \mathbb{D}^b_G(\tilde{X}) \)

\[
E_2^{u,v} = \text{Hom}^*(\mathcal{L}^{-r}, \mathcal{H}^v(p^* \mathcal{O}_s \otimes \chi^{-r})) \Rightarrow E_\infty^{u,v} = \text{Hom}^{u+v}(\mathcal{L}^{-r}, p^* \mathcal{O}_s \otimes \chi^{-r})
\]

it follows that

\[
\mathbb{C}[0] \cong \text{Hom}^*_{\mathbb{D}^b_G(\tilde{X})}(\mathcal{L}^{-r}, p^* \mathcal{O}_s \otimes \chi^{-r}) \\
\cong \text{Hom}^*_{\mathbb{D}^b_G(\tilde{X})}(\mathcal{L}^{-r}, \mathcal{H}^{-r}(p^* \mathcal{O}_s) \otimes \chi^{-r})[r] \\
\cong (\text{Hom}^*_{\mathbb{D}^b(\mathbb{P}^{n-1})}(\mathcal{O}(r), \Omega^r(r)) \otimes \chi^{\alpha_r} \otimes \chi^{-r})^G[r] \\
\cong (\mathbb{C}[-r] \otimes \chi^{\alpha_r-r})^G[r]
\]

where the last isomorphism is again due to Lemma 2.2.3. Comparing the first and last term of the above chain of isomorphisms, we get \( \mathbb{C} \cong (\chi^{\alpha_r-r})^G \) which implies \( \alpha_r = r \).

**Corollary 2.4.9.** Let \( n \geq m \) and \( \ell \in \{0, \ldots, m-1\} \). Let \( \lambda \geq 0 \) be the largest integer such that \( \ell + \lambda m \leq n - 1 \). Then

\[
\mathcal{H}^*(\Psi(\mathcal{O}_s \otimes \chi^{-\ell})) \cong i_{ss} \left( \bigoplus_{t=0}^{\lambda} \Omega^{\ell+tm}(\ell + tm)[\ell + tm] \right) .
\]

**Proof.** Since the (non-derived) functor \( q^G_* : \text{Coh}^G(\tilde{X}) \rightarrow \text{Coh}(\tilde{Y}) \) is exact, we have

\[
\mathcal{H}^*(\Psi(\mathcal{O}_s \otimes \chi^{-\ell})) \cong q^G_*(\mathcal{H}^{-r}(p^* \mathcal{O}_s) \otimes \chi^{-\ell})
\]

and the claim follows from Lemma 2.4.8. \( \square \)

### 2.4.3 On the cyclic cover

The morphism \( q : \tilde{X} \rightarrow \tilde{Y} = \tilde{X}/G \) is a cyclic cover branched over the divisor \( Z \). This geometric situation and the derived categories involved are studied in great detail in \([KP17]\). However, we will only need the following basic facts, all of which can be found in \([KP17, \text{Sect. 4.1}]\).

**Lemma 2.4.10.**

1. The sheaf of invariants \( q_*^G(\mathcal{O}_{\tilde{X}} \otimes \chi^{-1}) \) is a line bundle which we denote \( \mathcal{L}_Y^{-1} \in \text{Pic}(\tilde{Y}) \).
2. \( \mathcal{L}_Y^m \cong \mathcal{O}_Y(Z) \).
3. \( q_*^G(\mathcal{O}_{\tilde{X}} \otimes \chi^\alpha) \cong \mathcal{L}_Y^\alpha \) for \( \alpha \in \{-m + 1, \ldots, 0\} \).
4. $q^* \circ \text{triv}: \text{D}^b(\overline{Y}) \hookrightarrow \text{D}^b(\overline{X})$ is fully faithful, due to $q_*^G(\mathcal{O}_{\overline{X}}) \cong \mathcal{O}_{\overline{Y}}$.

5. $q^*(\text{triv}(\mathcal{L}^c)) \cong \mathcal{L}^c$ are isomorphic $G$-equivariant line bundles.

6. In particular, $\mathcal{L}_{\overline{Y}|Z} \cong \mathcal{L}_{\overline{X}|Z} \cong \mathcal{O}_{\nu}(-1)$.

**Corollary 2.4.11.** $\Psi(\mathcal{O}_X \otimes \chi^\alpha) \cong \mathcal{L}_{\overline{Y}}^{\alpha}$ for $\alpha \in \{-m+1, \ldots, 0\}$.

**Lemma 2.4.12.** The relative dualising sheaf of $q$: $\overline{X} \to \overline{Y} = \overline{X}/G$ is $\omega_q \cong \mathcal{O}_{\overline{X}}((m-1)Z)$.

**Proof.** Since the $G$-action on $W := \overline{X} \setminus Z$ is free, we have $\omega_{q|W} \cong \mathcal{O}_W$. Hence, $\omega_q \cong \mathcal{O}_{\overline{X}}(\alpha Z)$ for some $\alpha \in \mathbb{Z}$. We have $\text{Hom}(\mathcal{O}_Z, \mathcal{O}_{\overline{X}}) \cong \mathcal{O}_Z(Z)[-1] \cong j_*\mathcal{O}_v(-1)[-1]$, and hence

$$i_*\mathcal{O}_v(-1)[-1] \cong q_*j_*\mathcal{O}_v(-1)[-1] \cong q_*\text{Hom}(\mathcal{O}_Z, \mathcal{O}_{\overline{X}}) \cong q_*\text{Hom}(\mathcal{O}_Z, q^*\mathcal{O}_{\overline{X}}) \cong \text{Hom}(q_*\mathcal{O}_Z, \mathcal{L}^{-\alpha}) \text{ by Lemma 2.4.10(v)} \cong \text{Hom}(q_*\mathcal{O}_Z, \mathcal{L}^{-\alpha}) \text{ by Grothendieck duality} \cong \mathcal{O}_Z(Z) \otimes \mathcal{L}^{-\alpha}[-1] \cong i_*\mathcal{O}_v(-m+\alpha)[-1] \text{ by Lemma 2.4.10(ii)+(vi)}$$

and thus we conclude $\alpha = m - 1$. \hfill \Box

**Remark 2.4.13.** As an equivariant bundle, we have $\omega_q \cong \mathcal{L}^{m-1}_X \otimes \chi$, but we will not use this in the following.

**Corollary 2.4.14.** We have $\omega_{\overline{Y}|Z} \cong \mathcal{O}_{\nu}(m-n)$.

**Proof.** We have $\omega_{\overline{X}|Z} \cong \mathcal{O}_{\nu}(-n+1)$; compare Lemma 2.4.7. Furthermore, $\omega_{\overline{Y}|Z} \cong (q^*\omega_{\overline{Y}})|Z$. Hence,

$$\mathcal{O}_{\nu}(1 - m) \stackrel{2.4.12}{{\cong}} \omega_{q|Z} \cong \omega_{\overline{X}|Z} \otimes \omega_{\overline{Y}|Z} \cong \mathcal{O}_{\nu}(1 - n) \otimes \omega_{\overline{Y}|Z}.$$

\hfill \Box

### 2.4.4 The case $m \geq n$

Throughout this subsection, let $m \geq n$.

**Proposition 2.4.15.**

1. If $m > n$, then the functor $\Xi_\alpha$ is fully faithful for any $\alpha \in \mathbb{Z}/m\mathbb{Z}$.

2. Let $m - n \geq 2$ and $\alpha \neq \beta \in \mathbb{Z}/m\mathbb{Z}$. Then

$$\Xi_\beta^R\Xi_\alpha = 0 \iff \alpha - \beta \in \{n - m + 1, n - m + 2, \ldots, -1\}.$$

**Proof.** Recall that $\Xi_\beta = (a_* \circ \text{triv}(\_)) \otimes \chi^\beta: \text{D}^b(S) \to \text{D}^b_G(X)$. Hence, the right-adjoint of $\Xi_\beta$ is given by $\Xi_\beta^R \cong (a'_* (\_ \otimes \chi^{-\beta})^G)$. By [AC12, Thm. 1.4 & Sect. 1.20],

$$\Xi_\beta^R\Xi_\alpha \cong (a'_* (\_ \otimes \chi^{\alpha-\beta})^G) \cong (\_ \otimes \wedge^* N \otimes \chi^{\alpha-\beta})^G \cong (\_ \otimes (\wedge^* N \otimes \chi^{\alpha-\beta})^G$$

82
where, by Condition 2.3.1, the $G$-action on $\wedge^N$ is given by $\chi^{-\ell}$. We see that $(\wedge^N)^G \cong \wedge^N[0] \cong \mathcal{O}_S[0]$; here we use that $m > n$. This shows that, in the case $\alpha = \beta$, we have $\Xi^R\Xi\cong\text{id}$ which proves (i). Furthermore, since the characters occurring in $\wedge^N$ are $\chi^0$, $\chi^{-1}, \ldots, \chi^{-n}$, we obtain (ii) from

$\Xi^R\Xi\neq0 \iff (\wedge^N \otimes \chi^{0-\beta})^G \neq 0 \iff \Xi \in \{\alpha - \beta, \alpha - \beta - 1, \ldots, \alpha - \beta - n\}$, i.e.

$\Xi^R\Xi = 0 \iff \alpha - \beta \in \{n+1, \ldots, m-1\} = \{n-m+1, n-m+2, \ldots, -1\}$.

**Corollary 2.4.16.** For $m > n$, there is a semi-orthogonal decomposition

$$D^b_G(X) = \langle \Xi_{n-m}(D^b(S)), \Xi_{n-m+1}(D^b(S)), \ldots, \Xi_1(D^b(S)), A \rangle,$$

where $A = \perp \langle \Xi_{n-m}(D^b(S)), \Xi_{n-m+1}(D^b(S)), \ldots, \Xi_1(D^b(S)) \rangle$.

**Proposition 2.4.17.** The functor $\Phi = p_*q^*\text{triv}: D^b(\tilde{Y}) \to D^b_G(X)$ is fully faithful.

**Proof.** By [Huy06, Prop. 7.1], we only need to show for $x, y \in \tilde{Y}$ that

$$\text{Hom}^i_{D^b_G(X)}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_y)) = \begin{cases} \mathbb{C} & \text{if } x = y \text{ and } i = 0 \\ 0 & \text{if } x \neq y \text{ or } i \notin [0, \dim X]. \end{cases}$$

By Proposition 2.3.2, $\Phi(\mathcal{O}_x) = \mathcal{O}_\xi$ for some $G$-cluster $\xi$. Hence,

$$\text{Hom}^0_{D^b_G(X)}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x)) \cong H^0(\mathcal{O}_\xi)^G \cong \mathbb{C}.$$ 

Furthermore, since $\Phi(\mathcal{O}_x)$ is a sheaf, the complex $\text{Hom}^*_{\xi}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x))$ is concentrated in degrees $0, \ldots, \dim(X)$. It remains to prove the orthogonality for $x \neq y$. If $g(x) \neq g(y)$, the corresponding $G$-clusters are supported on different orbits. Hence, their structure sheaves are orthogonal. If $g(x) = g(y)$ but $x \neq y$, the orthogonality was shown in Lemma 2.3.3. \qed

**Lemma 2.4.18.** The functor $\Phi$ factors through $A$.

**Proof.** By Corollary 2.4.16, this statement is equivalent to $\Phi^R\Xi = 0$ for $\alpha \in \{n-m, \ldots, -1\}$, where $\Phi^R: D^b_G(X) \to D^b(\tilde{Y})$ is the right adjoint of $\Phi$. Since the composition $\Phi^R\Xi$ is a Fourier–Mukai transform, it is sufficient to test the vanishing on skyscraper sheaves of points; see [Kuz06, Sect. 2.2]. So we have to prove that

$$\Phi^R\Xi(\mathcal{O}_s) \cong \Phi^R(\mathcal{O}_s \otimes \chi^\alpha) = 0$$

for every $s \in S$ and every $\alpha \in \{n-m, \ldots, -1\}$. We have $\Phi^R \cong q^G_*p^!$; recall that $q^G_*$ stands for $(_-)^G \circ q_*$. By Lemma 2.4.7 together with Lemma 2.4.8, we have

$$\mathcal{H}^{-r}(p^!\mathcal{O}_s) \cong i_{ss}(\Omega^r(r+1-n) \otimes \chi^{r+1-n})$$

where the non-vanishing cohomologies occur for $r \in \{0, \ldots, n-1\}$. Thus, the linearisations of the cohomologies of $p^!(\mathcal{O}_s \otimes \chi^\alpha)$ are given by the characters $\chi^\gamma$ for $\gamma \in \{\alpha + 1 - n, \ldots, \alpha\}$. We see that, for $\alpha \in \{n-m, \ldots, -1\}$, the trivial character does not occur in $\mathcal{H}^*(p^!\mathcal{O}_s \otimes \chi^\alpha)$. This implies that $q_*p^!(\mathcal{O}_s \otimes \chi^\alpha)$ has vanishing $G$-invariants. \qed
We denote by \( \mathcal{B} \subset \text{D}^b_G(X) \) the full subcategory generated by the admissible subcategories \( \Xi_\alpha(\text{D}^b(S)) \) for \( \alpha \in \{n-m, \ldots, -1\} \) and \( \Phi(\text{D}^b(Y)) \). By the above, these admissible subcategories actually form a semi-orthogonal decomposition

\[
\mathcal{B} = \langle \Xi_{n-m}(\text{D}^b(S)), \Xi_{n-m+1}(\text{D}^b(S)), \ldots, \Xi_{-1}(\text{D}^b(S)) \rangle.
\]

**Proposition 2.4.19.** We have the (essential) equalities \( \mathcal{B} = \text{D}^b_G(X) \) and \( \Phi(\text{D}^b(Y)) = \mathcal{A} \).

For the proof, we need the following

**Lemma 2.4.20.** We have \( p_* \mathcal{L}^r_X \otimes \chi^{-\lambda} \in \mathcal{B} \) for \( r \in \mathbb{Z} \) and \( \lambda \in \{0, \ldots, m-n\} \).

**Proof.** By Lemma 2.4.10, \( \mathcal{L}^r_X \cong q^*(\text{triv}(\mathcal{L}^r_Y)) \). Hence,

\[
p_* (\mathcal{L}^r_X) \cong p_*q^*(\text{triv}(\mathcal{L}^r_Y)) = \Phi(\mathcal{L}^r_Y) \in \Phi(\text{D}^b(Y)) \subset \mathcal{B}
\]

which proves the assertion for \( \lambda = 0 \). We now proceed by induction over \( \lambda \). Tensoring (2.11) by \( \mathcal{L}^r_X \otimes \chi^{-\lambda} \) and applying \( p_* \), we get the exact triangle

\[
p_* \mathcal{L}^{r-1}_X \otimes \chi^{-(\lambda-1)} \rightarrow p_* \mathcal{L}^r_X \otimes \chi^{-\lambda} \rightarrow p_* j_* \mathcal{O}_Z(-r) \otimes \chi^{-\lambda} \rightarrow \tag{2.12}
\]

where \( \mathcal{O}_Z(-r) \) carries the trivial \( G \)-action.

The first term of the triangle is an object of \( \mathcal{B} \) by induction. Furthermore, by diagram (2.8), we have \( p_* j_* \mathcal{O}_Z(-r) \cong a_* \nu_* \mathcal{O}_Z(-r) \). Hence, the third term of (2.12) is an object of \( a_* \text{D}^b_G(S) \otimes \chi^{-\lambda} = \Xi_{-\lambda}(\text{D}^b(S)) \subset \mathcal{B} \). Thus, also the middle term is an object of \( \mathcal{B} \) which gives the assertion. \( \square \)

**Proof of Proposition 2.4.19.** The second assertion follows from the first one since, if \( \mathcal{B} = \text{D}^b_G(X) \) holds, both, \( \Phi(\text{D}^b(Y)) \) and \( \mathcal{A} \), are given by the left-orthogonal complement of

\[
\langle \Xi_{n-m}(\text{D}^b(S)), \Xi_{n-m+1}(\text{D}^b(S)), \ldots, \Xi_{-1}(\text{D}^b(S)) \rangle
\]

in \( \text{D}^b_G(X) \).

The subcategories \( \Xi_\alpha(\text{D}^b(S)) \) and \( \Phi(\text{D}^b(Y)) \) of \( \text{D}^b_G(X) \) are \( Y \)-linear by Lemma 2.4.4. Hence, for the equality \( \mathcal{B} = \text{D}^b_G(X) \) it suffices to show that

\[
\mathcal{O}_X \otimes \chi^\ell \in \mathcal{B} = \langle \text{D}^b(S) \otimes \chi^{n-m}, \ldots, \text{D}^b(S) \otimes \chi^{-1}, \Phi(\text{D}^b(Y)) \rangle
\]

for every \( \ell \in \mathbb{Z}/m\mathbb{Z} \); see Lemma 2.4.3. Combining Lemma 2.4.10 and Lemma 2.4.6, we see that

\[
\Phi(\mathcal{L}^\ell_Y) \cong p_*(q^*(\text{triv}(\mathcal{L}^\ell_Y)) \cong \mathcal{O}_X \otimes \chi^\ell \quad \text{for} \quad \ell = 0, \ldots, n-1.
\]

In particular, \( \mathcal{O}_X \otimes \chi^\ell \in \Phi(\text{D}^b(Y)) \subset \mathcal{B} \) for \( \ell = 0, \ldots, n-1 \). Setting \( r = 0 \) in the previous lemma, we find that also \( \mathcal{O}_X \otimes \chi^\ell \) for \( \ell = n-m, \ldots, -1 \) is an object of \( \mathcal{B} \). \( \square \)

Combining the results of this subsection gives Theorem 2.4.1(i).
2.4.5 The case \( n \geq m \)

Throughout this subsection, let \( n \geq m \).

**Proposition 2.4.21.** Let \( n > m \). Then the functors \( \Theta_\beta : \mathcal{D}^b(S) \to \mathcal{D}^b(\tilde{Y}) \) are fully faithful for every \( \beta \in \mathbb{Z} \) and there is a semi-orthogonal decomposition

\[
\mathcal{D}^b(\tilde{Y}) = \langle \mathcal{C}(m - n), \mathcal{C}(m - n + 1), \ldots, \mathcal{C}(-1), \mathcal{D} \rangle
\]

where \( \mathcal{C}(\ell) := \Theta_\ell(\mathcal{D}^b(S)) = i_* \nu^* \mathcal{D}^b(S) \otimes \mathcal{O}_{\tilde{Y}}(\ell) \) and

\[
\mathcal{D} = \{ E \in \mathcal{D}^b(\tilde{Y}) \mid i^* E \in \langle \nu^* \mathcal{D}^b(S) \otimes \mathcal{O}_{\tilde{Y}}(m - n), \ldots, \nu^* \mathcal{D}^b(S) \otimes \mathcal{O}_{\tilde{Y}}(-1) \rangle \}
\]

\[
= \{ E \in \mathcal{D}^b(\tilde{Y}) \mid i^* E \in \langle \nu^* \mathcal{D}^b(S), \ldots, \nu^* \mathcal{D}^b(S) \otimes \mathcal{O}_{\tilde{Y}}(m - 1) \rangle \}.
\]

**Proof.** This follows from [Kuz08, Thm. 1]. However, for convenience, we provide a proof for our special case. By construction, \( \Theta_\beta^R \cong \nu_* M_{\mathcal{O}_\nu(-\beta)} i^! \). We start with the standard exact triangle of functors \( \text{id} \to i^! i_* \to \mathcal{M}_{\mathcal{O}_Z(Z)[-1]} \to (\text{see e.g. [Huy06, Cor. 11.4]}) \). By Lemma 2.4.10, \( \mathcal{O}_Z(Z) \cong \mathcal{O}_\nu(-m) \), and thus the above triangle induces for any \( \alpha, \beta \in \mathbb{Z} \)

\[
\nu_* M_{\mathcal{O}_\nu(\alpha - \beta)} \nu^* \to \Theta_\beta^R \Theta_\alpha \to \nu_* M_{\mathcal{O}_\nu(\alpha - \beta - m)} \nu^* \to .
\]

By projection formula, we can rewrite this as

\[
(\_ \otimes \nu_* \mathcal{O}_\nu(\alpha - \beta) \to \Theta_\beta^R \Theta_\alpha \to (\_ \otimes \nu_* \mathcal{O}_\nu(\alpha - \beta - m) \to .
\]

Now, \( \nu_* \mathcal{O}_\nu \cong \mathcal{O}_S \) and \( \nu_* \mathcal{O}_\nu(\gamma) = 0 \) for \( \gamma \in \{-n + 1, \ldots, -1\} \). Hence, \( \Theta_\beta^R \Theta_\alpha \cong \text{id} \) and \( \Theta_\beta^R \Theta_\alpha = 0 \) if \( \alpha - \beta \in \{-m + 1, \ldots, -1\} \). Therefore, we get a semi-orthogonal decomposition

\[
\mathcal{D}^b(\tilde{Y}) = \langle \mathcal{C}(m - n), \mathcal{C}(m - n + 1), \ldots, \mathcal{C}(-1), \mathcal{D} \rangle.
\]

The description of the left-orthogonal \( \mathcal{D} \) follows by the adjunction \( i^* \dashv i_* \). \( \square \)

**Lemma 2.4.22.** The functor \( \Psi : \mathcal{D}^b_G(X) \to \mathcal{D}^b(\tilde{Y}) \) factors through \( \mathcal{D} \).

**Proof.** By Lemma 2.4.3, the equivariant bundles \( \mathcal{O}_X, \mathcal{O}_X \otimes \chi, \ldots, \mathcal{O}_X \otimes \chi^{m-1} \) generate \( \mathcal{D}^b_G(X) \) over \( \mathcal{D}^\text{perf}(Y) \), and therefore so do the bundles \( \mathcal{O}_X \otimes \chi^{-m+1}, \ldots, \mathcal{O}_X \otimes \chi^{-1}, \mathcal{O}_X \) obtained by twisting with \( \chi^{1-m} \). Hence, it is sufficient to prove that \( \Psi(\mathcal{O}_X \otimes \chi^\alpha) \in \mathcal{D} \) for \( \alpha \in \{-m + 1, \ldots, 0\} \) as \( \Psi \) and \( \mathcal{D} \) are \( Y \)-linear; see Lemma 2.4.4.

Indeed, by Lemma 2.4.10 we have \( i^* L^\alpha_Y = L^\alpha_{\tilde{Y}|Z} \cong \mathcal{O}_\nu(-\alpha) \), hence

\[
\Psi(\mathcal{O}_X \otimes \chi^\alpha) \cong q_\nu^G(\mathcal{O}_X \otimes \chi^\alpha) \cong L^\alpha_{\tilde{Y}} \in \mathcal{D} \quad \text{for } \alpha \in \{-m + 1, \ldots, 0\}. \]

\( \square \)

**Proposition 2.4.23.** The functor \( \Psi : \mathcal{D}^b_G(X) \to \mathcal{D}^b(\tilde{Y}) \) is fully faithful.

**Proof.** We first observe that \( V := \mathcal{O}_X \otimes \mathbb{C}[G] = \mathcal{O}_X \otimes (\chi^0 \oplus \cdots \oplus \chi^{m-1}) \) is a relative tilting bundle for \( \mathcal{D}^b_G(X) \) over \( \mathcal{D}^\text{perf}(Y) \); see Lemma 2.4.3.
For the fully faithfulness, we follow Lemma 2.2.10. So we need to show that \( \Psi \) induces an isomorphism \( \Lambda V = \pi^G \text{Hom}(V, V) \sim \varrho_* \text{Hom}(\Psi(V), \Psi(V)) \). In turn, it suffices to consider the direct summands of \( V \). Thus, let \( \alpha, \beta \in \{-m + 1, \ldots, 0\} \) and compute

\[
\pi^G \text{Hom}^*(\mathcal{O}_X \otimes \chi^\alpha, \mathcal{O}_X \otimes \chi^\beta) \cong \pi^G \text{Hom}^*(\mathcal{O}_X \otimes \chi^{\alpha+n-1} \otimes \chi^{1-n}, \mathcal{O}_X \otimes \chi^\beta)
\]

\[
\cong \pi^G \text{Hom}^*(p_* \mathcal{L}_X^\alpha \otimes \omega_p, \mathcal{O}_X \otimes \chi^\beta)
\]

\[
\cong \pi^G p_* \text{Hom}_X^*(\mathcal{L}_X^\alpha \otimes \omega_p, p^! \mathcal{O}_X \otimes \chi^\beta)
\]

\[
\cong \pi^G p_* \text{Hom}_X^*(\mathcal{L}_X^\alpha, \mathcal{O}_X \otimes \chi^\beta)
\]

\[
\cong \varrho_* q^*_G \text{Hom}^*_Y(\mathcal{O}^\alpha \otimes \chi^\beta).
\]

\[
\cong \varrho_* \text{Hom}^*_Y(\mathcal{O}^\alpha \otimes \chi^\beta).\]

We denote by \( \mathcal{E} \subset \mathcal{D}^b(\mathcal{Y}) \) the full subcategory generated by the admissible subcategories \( \Psi(\mathcal{D}^b_G(X)) \) and \( \Theta(S) = \mathcal{O}_Y(\ell) \) for \( \ell \in \{m-n, \ldots, -1\} \). By the above, these admissible subcategories actually form a semi-orthogonal decomposition

\[
\mathcal{E} = \langle \Theta_{m-n}(\mathcal{D}^b(S)), \ldots, \Theta_{-1}(\mathcal{D}^b(S)), \Psi(\mathcal{D}^b_G(X)) \rangle \subset \mathcal{D}^b(\mathcal{Y}).
\]

**Proposition 2.4.24.** We have the (essential) equalities \( \mathcal{E} = \mathcal{D}^b(\mathcal{Y}) \) and \( \Psi(\mathcal{D}^b_G(X)) = \mathcal{D} \).

**Proof.** Analogously to Proposition 2.4.19, it is sufficient to prove the equality \( \mathcal{E} = \mathcal{D}^b(\mathcal{Y}) \). As \( \mathcal{E} \) is constructed from images of fully faithful FM transforms (which have both adjoints), it is admissible in \( \mathcal{D}^b(\mathcal{Y}) \). Therefore, it suffices to show that \( \mathcal{E} \) contains a spanning class for \( \mathcal{D}^b(\mathcal{Y}) \). Moreover, because all functors and categories involved are \( \mathcal{Y} \)-linear, it suffices to prove that the relative spanning class \( \mathcal{S} \) of Lemma 2.4.5 is contained in \( \mathcal{E} \).

We already know that \( \mathcal{O}_Y \cong \Psi(\mathcal{O}_X) \in \Psi(\mathcal{D}^b_G(X)) \subset \mathcal{E} \). By Corollary 2.2.4, we get for \( s \in S \) and \( r \in \{m, \ldots, n-1\} \)

\[
i_{ss} \Omega^r(\ell) \in \langle \Theta_{m-n}(\mathcal{D}^b(S)), \ldots, \Theta_{-1}(\mathcal{D}^b(S)) \rangle \subset \mathcal{E}.
\]

By Corollary 2.4.9, we have, for \( \ell \in \{0, \ldots, m-1\} \), an exact triangle

\[
E \to \Psi(\mathcal{O}_s \otimes \chi^{-\ell}) \to i_{ss} \Omega^r(\ell)[\ell] \to
\]

where \( E \) is an object in the triangulated category spanned by \( i_{ss} \Omega^r(\ell) \) for \( r \in \{m, \ldots, n-1\} \). In particular, the first two terms of the exact triangle are objects in \( \mathcal{E} \). Hence also \( i_{ss} \Omega^r(\ell) \in \mathcal{E} \) for \( \ell \in \{0, \ldots, m-1\} \).

Combining the results of this subsection gives Theorem 2.4.1(ii).

**2.4.6 The case \( m = n \): spherical twists and induced tensor products**

Throughout this section, let \( m = n \). In this case, both functors \( \Phi \) and \( \Psi \) are equivalences. In this subsection, we will show that the functors \( \Theta_\beta \) and \( \Xi_\alpha \), which were fully faithful in the cases \( n > m \) and \( m > n \), respectively, are now spherical. Furthermore, the spherical twists along these functors allow to describe the transfer of the tensor structure from one side of the derived McKay correspondence to the other. We set \( \Theta := \Theta_0 \) and \( \Xi := \Xi_0 \).
**Proposition 2.4.25.** For every \( \alpha \in \mathbb{Z}/m\mathbb{Z} \), the functor \( \Xi_\alpha : \mathcal{D}^b(S) \to \mathcal{D}^b_G(X) \) is a split spherical functor with cotwist \( M_{\omega S/X}[-n] \).

**Proof.** Since \( \Xi \equiv M_\alpha \Xi \), it is sufficient to prove the assertion for \( \alpha = 0 \); see Lemma 2.2.14. Following the proof of Proposition 2.4.15, we have \( \Xi^R \Xi \cong (\_\_ \otimes (\wedge^* N)^G \ where \ G \ acts \ on \ \wedge^\ell N \ by \ \chi^{-\ell} \. \) From rank \( N = n = m = \text{ord} \chi \), we get

\[
(\wedge^* N)^G \cong \mathcal{O}_S[0] \oplus \text{det } N[-n] \cong \mathcal{O}_S[0] \oplus \omega_{S/X}[-n].
\]

Hence, \( \Xi^R \Xi \cong \text{id} \oplus C \) with \( C := M_{\omega S/X}[-n] \). Moreover, we have \( a^1 \cong C a^* \) which implies \( \Xi^R \cong C \Xi^L \). \( \square \)

We introduce autoequivalences \( M_\mathcal{L} : \mathcal{D}^b(\bar{Y}) \to \mathcal{D}^b(\bar{Y}) \) and \( M_\chi : \mathcal{D}^b_G(X) \to \mathcal{D}^b_G(X) \) given by the tensor products with the line bundle \( \mathcal{L}_\mathcal{V} \) and the character \( \chi \), respectively.

**Theorem 2.4.26.** There are the following relations between functors:

1. \( \Psi^{-1} \cong M_\chi \Phi M_{\mathcal{L}^{-1}} \);
2. \( \Psi \Xi \cong \Theta \), in particular, the functors \( \Theta_\beta \) are spherical too;
3. \( T_\Theta \cong \Psi T_\Xi \Psi^{-1} \);
4. \( \Psi^{-1} M_{\mathcal{L}} \Psi \cong M_\chi T_\Xi \) and \( \Psi^{-1} M_{\mathcal{L}^{-1}} \Psi \cong T_\Xi^{-1} M_\chi^{-1} \).

**Proof.** To verify (i), note that

\[
\Psi^{-1} \cong \Psi^L \cong p^* q^* \overset{2.4.7}{\cong} p_* M_{\mathcal{L}^{-1}_X} q^* \overset{2.4.10}{\cong} M_\chi p_* q^* M_{\mathcal{L}^{-1}} \cong M_\chi \Phi M_{\mathcal{L}^{-1}}.
\]

For (ii), first note that, since the \( G \)-action on \( Z \subset \bar{X} \) is trivial, we have
\[
\Theta \cong i_* \nu^* \cong q_* j_* \nu^* \cong q_*^G j_* \nu^* \text{ triv}.
\]

Hence, the base change morphism \( \vartheta : p^* a_* \to j_* \nu^* \) induces a morphism of functors
\[
\vartheta : \Psi \Xi \cong q_*^G p_* a_* \text{ triv} \to q_*^G j_* \nu^* \text{ triv} \cong \Theta
\]

which in turn is induced by a morphism between the Fourier–Mukai kernels; see [Kuz06, Sect. 2.4]. Hence, it is sufficient to show that \( \vartheta \) induces an isomorphism \( \Psi \Xi(\mathcal{O}_s) \cong \Theta(\mathcal{O}_s) \) for every \( s \in S \); see [Kuz06, Sect. 2.2]. The morphism \( \vartheta \) induces an isomorphism on degree zero cohomology \( L_0^p a_* (\mathcal{O}_s) \cong L_{p^{-1}(a(s))} \cong j_* L_0^0 \nu^* (\mathcal{O}_s) \). But there are no cohomologies in non-zero degrees for \( j_* \nu^* \) since \( \nu \) is flat and \( j \) a closed embedding. Furthermore, the non-zero cohomologies of \( p^* a_* \) vanish after taking invariants; see Corollary 2.4.9. Hence, \( \vartheta (\mathcal{O}_s) \) is indeed an isomorphism.

The second assertion of (ii) and (iii) are direct consequences of Proposition 2.4.25 and the formula \( \Psi \Xi \cong \Theta \); see Lemma 2.2.14.

For (iv), it is sufficient to prove the second relation, and we employ Corollary 2.2.12 with \( L_1 = \mathcal{O}_X \otimes \chi^\alpha \); see also Lemma 2.4.3. Recall that \( T_\Xi^{-1} = \text{cone}(\text{id} \to \Xi \Xi^L)[-1] \), and \( \Xi^L \cong (\_\_)^G a^* \). For \( 1 \neq \alpha \in \mathbb{Z}/n\mathbb{Z} \), we get

\[
\Xi^L M_{\mathcal{L}^{-1}} (\mathcal{O}_X \otimes \chi^\alpha) \cong (\mathcal{O}_S \otimes \chi^{\alpha-1})^G \cong 0.
\]
For example, we can deduce the formula $M_{X \to Y}(O_Y \otimes \chi) \cong O_X \otimes \chi^{-1}$. We have $\Xi^L(O_X) = O_S$. Therefore, $\Xi^L(O_X) \cong a_0 \mathcal{O}_S$ and $T_\Xi(O_X) \cong \mathcal{I}_S$. In summary,

$$T_\Xi^{-1} M_{X \to Y}(O_Y \otimes \chi) \cong \begin{cases} O_X \otimes \chi & \text{for } \alpha \neq 1, \\ \mathcal{I}_S & \text{for } \alpha = 1. \end{cases}$$

On the other hand, for $\alpha \in \{-n+1,\ldots,0\}$, we have $\Psi(O_X \otimes \chi) \cong \mathcal{L}^\alpha$; see Corollary 2.4.11. Hence, we have

$$\Psi^{-1} M_{\mathcal{L}^{-1}} \Psi(O_X \otimes \chi) \cong O_X \otimes \chi^{-1} \text{ for } \alpha \in \{-n+2,\ldots,0\}.$$ 

For $\alpha = -n+1$, we use (i) to get

$$\Psi^{-1} M_{\mathcal{L}^{-1}} \Psi(O_X \otimes \chi^{-1-n}) \cong \Psi^{-1}(\mathcal{L}_Y^{-n}) \cong M_\chi \Phi(\mathcal{L}_Y^{-1}) \cong p_*(\mathcal{L}_X^{-1} \otimes \chi) \cong \mathcal{I}_S$$

where we get the last isomorphism by applying $p_*$ to the exact sequence (2.11).

Therefore, we obtain isomorphisms

$$\kappa_\alpha : F_1(L_\alpha) := T_\Xi^{-1} M_{X \to Y}(O_X \otimes \chi) \xrightarrow{\sim} F_2(L_\alpha) := \Psi^{-1} M_{\mathcal{L}^{-1}} \Psi(O_X \otimes \chi)$$

for every $\alpha \in \mathbb{Z}/n\mathbb{Z}$.

Finally, we have to check that the isomorphisms $\kappa_\alpha$ can be chosen is such a way that they form an isomorphism of functors $\kappa : F_{1,V}(L_{0,\ldots,0}) \xrightarrow{\sim} F_{2,V}(L_{0,\ldots,0})$ over every open set $V \subset Y$. Let $U := Y \setminus S \subset Y$ the open complement of the singular locus. We claim that $F_{1,U} \cong M_\chi^{-1} \cong F_{2,U}$. This is clear for $F_2 = T_\Xi^{-1} M_\chi^{-1}$. Furthermore, the map $p : \tilde{X} \to X$ is an isomorphism and $q : \tilde{X} \to \tilde{Y}$ is a free quotient when restricted to $W := \pi^{-1}(U)$. Since also $\mathcal{L}_\tilde{X} = q_*^\phi (O_{\tilde{X}} \otimes \chi)$, we get $\Psi_U \cong M_\chi^{-1} |_U$.

Hence, over $W$, the $\kappa_{i|W}$ can be chosen functorially. By the above computations, each $\kappa_{i|W}$ is given by a section of the trivial line bundle. As $S$ has codimension at least 2 in $X$, the sections $\kappa_{i|W}$ over $W$ uniquely extend to sections $\kappa_i$ over $X$. The commutativity of the diagrams relevant for the functoriality now follows from the commutativity of the diagrams restricted to the dense subset $W$. 

The relations of Theorem 2.4.26 allow to transfer structures between $D^b(\tilde{Y})$ and $D^b_G(X)$. For example, we can deduce the formula $\Psi M_{X^{-1}} \Psi^{-1} \cong T_\Theta M_{\mathcal{L}^{-1}}$. Since $O_X \otimes \chi$ for $\alpha \in \{-(n-1),\ldots,0\}$ form a relative generator of $D^b_G(X)$, their images $\mathcal{L}^\alpha$ under $\Psi$ do as well. Hence, at least theoretically, our formulas give a complete description of the tensor products induced by $\Psi$ (and also $\Phi$) on both sides.

Note that $\Phi$ and $\Psi$ are both equivalences, but not inverse to each other. Hence, they induce non-trivial autoequivalences $\Psi \Phi \in \text{Aut}(D^b_G(X))$ and $\Phi \Psi \in \text{Aut}(D^b(\tilde{Y}))$. Considering the setup of the McKay correspondence as a flop of orbifolds as in diagram (2.1), it makes sense to call them flop-flop autoequivalences. These kinds of autoequivalences were widely studied for fops of varieties; see [Tod07], [BB15], [DW16], [DW15], [ADM15]. The general picture seems to be that the flop-flop autoequivalences can be expressed via spherical and $\mathbb{P}$-twists induced by functors naturally associated to the centres of the fops. This picture is called the 'flop-flop-twist' principle; see [ADM15]. The following can be seen as the first instance of an orbifold 'flop-flop-twist' principle which we expect to hold in greater generality.

**Corollary 2.4.27.** $\Psi \Phi \cong T_\Theta M_{\mathcal{L}^{-n}} \cong T_\Theta M_{\mathcal{O}_Y(-Z)}$. 

88
Remark 2.4.28. Let us assume $m = n = 2$ so that $\chi^{-1} = \chi$. Then, for every $k \in \mathbb{N}$, we get
\[
\Phi(L^{-k}) \cong I_S^k \otimes \chi^k
\]  
where $I_S^k$ denotes the $k$-th power of the ideal sheaf of the fixed point locus. Indeed,
\[
\Phi(L^{-k}) \cong M_\chi \Psi^{-1}(L^{-k-1}) \cong M_\chi(\Psi^{-1} M_{L^{-1}} \Psi)^k(L^{-1}) \cong M_\chi(\Psi^{-1} M_{L^{-1}} \Psi)^k(O \otimes \chi) \cong (M_\chi T_\Xi^{-1})^k(O_X) \cong I_S^k \otimes \chi^k.
\]
The last isomorphism follows inductively using the short exact sequences
\[0 \to I_S^{k+1} \to I_S^k \to I_S^k/I_S^{k+1} \to 0\]
and the fact that the natural action of $\mu_2$ on $I_S^k/I_S^{k+1}$ is given by $\chi^k$. Let now $S$ be a surface and $X = S^2$ with $\mu_2$ acting by permutation of the factors. Then $\tilde{Y} = S^{[2]}$ is the Hilbert scheme of two points and $\mathcal{L}_\tilde{Y}$ is the square root of the boundary divisor $Z$ parametrising double points. For a vector bundle $F$ on $S$ of rank $r$, we have
\[\det F^{[2]} \cong \mathcal{L}_\tilde{Y}^{-r} \otimes \mathcal{D}_{\det F}\]
where $F^{[2]}$ denotes the tautological rank $2r$ bundle induced by $F$ and, for $L \in \text{Pic} S$, we put $\mathcal{D}_L := g^* \pi_*(L \otimes L)^G \in \text{Pic} S^{[2]}$. Hence, by the $O_\gamma$-linearity of $\Phi$, formula (2.13) recovers the $n = 2$ case of [Sca15, Thm. 1.8].

2.5 Categorical resolutions

2.5.1 General definitions

Recall from [Kuz08] that a categorical resolution of a triangulated category $\mathcal{T}$ is a smooth triangulated category $\tilde{\mathcal{T}}$ together with a pair of functors $P_*: \tilde{\mathcal{T}} \to \mathcal{T}$ and $P^*: \mathcal{T}_{\text{perf}} \to \tilde{\mathcal{T}}$ such that $P^*$ is left adjoint to $P_*$ on $\mathcal{T}_{\text{perf}}$ and the natural morphism of functors $\text{id}_{\mathcal{T}_{\text{perf}}} \to P_* P^*$ is an isomorphism. Here, $\mathcal{T}_{\text{perf}}$ is the triangulated category of perfect objects in $\mathcal{T}$. Moreover, a categorical resolution $(\tilde{\mathcal{T}}, P_*, P^*)$ is weakly crepant if the functor $P^*$ is also right adjoint to $P_*$ on $\mathcal{T}_{\text{perf}}$.

For the notion of smoothness of a triangulated category see e.g. [KL15]. For us it is sufficient to notice that every admissible subcategory of $\mathcal{D}^b(Z)$ for some smooth variety $Z$ is smooth. In fact, we will always consider categorical resolutions of $\mathcal{D}^b(Y)$, for some variety $Y$ with rational Gorenstein singularities, inside $\mathcal{D}^b(\tilde{Y})$ for some fixed (geometric) resolution of singularities $\varrho: \tilde{Y} \to Y$. By this we mean an admissible subcategory $\tilde{\mathcal{T}} \subset \mathcal{D}^b(\tilde{Y})$ such that $\varrho^*: \mathcal{D}^b(\tilde{Y}) \to \mathcal{D}^b(Y)$ factorises through $\tilde{\mathcal{T}}$.

By Grothendieck duality, we get a canonical isomorphism $\mathcal{O}_Y \cong \varrho_* \mathcal{O}_\tilde{Y} \cong \varrho_* \omega_\varrho$. This induces a global section $s$ of $\omega_\varrho$, unique up to scalar multiplication by $\mathcal{O}_Y(\tilde{Y})^\times$, and hence a morphism of functors
\[t := \varrho_* (\_ \otimes s): \varrho_* \to \varrho_* \mathcal{O}_\tilde{Y}.\]
Since this morphism can be found between the corresponding Fourier–Mukai kernels, we may define the cone of functors \( \varrho_* := \text{cone}(t) : \mathcal{D}^b(\tilde{Y}) \to \mathcal{D}^b(Y) \).

**Definition 2.5.1.** The weakly crepant neighbourhood of \( Y \) inside \( \mathcal{D}^b(Y) \) is the full triangulated subcategory
\[
\text{WC}(\varrho) := \ker(\varrho_+) \subset \mathcal{D}^b(\tilde{Y}).
\]

**Proposition 2.5.2.** If \( \text{WC}(\varrho) \) is a smooth category (which is the case if it is an admissible subcategory of \( \mathcal{D}^b(\tilde{Y}) \)), it is a categorical weakly crepant resolution of singularities.

**Proof.** By adjunction formula, \( t\varrho^* : \varrho_*\varrho^* \to \varrho_+\varrho^* \) is an isomorphism. Hence, \( \varrho_+\varrho^* = 0 \) and \( \varrho^* : \mathcal{D}^\text{perf}(Y) \to \mathcal{D}^b(\tilde{Y}) \) factors through \( \text{WC}(\varrho) \). By definition, \( \varrho_+ \) is the left adjoint to \( \varrho^* \). Since \( \varrho_* \) and \( \varrho_+ \) agree on \( \text{WC}(\varrho) \), we also have the adjunction \( \varrho_* \dashv \varrho^* \) on \( \text{WC}(\varrho) \). \( \square \)

**Remark 2.5.3.** We think of \( \text{WC}(\varrho) \) as the biggest weakly crepant categorical resolution inside the derived category \( \mathcal{D}^b(\tilde{Y}) \) of a given geometric resolution \( \varrho : \tilde{Y} \to Y \). The only thing that prevents us from turning this intuition into a statement is the possibility that, for a given weakly crepant resolution \( T \subset \mathcal{D}^b(\tilde{Y}) \), there might be an isomorphism \( \varrho_*|_T \cong \varrho_+|_T \) which is not the restriction of \( t \) (up to scalars).

### 2.5.2 The weakly crepant neighbourhood in the cyclic setup

In the case of the resolution of the cyclic quotient singularities discussed in the earlier sections, \( \text{WC}(\varrho) \) is indeed a categorical resolution by the following result. We use the notation of Section 2.3; recall \( G = \mu_m \).

**Theorem 2.5.4.** Let \( Y = X/G, \; \varrho : \tilde{Y} \to Y \) and \( i : Z = \varrho^{-1}(S) \to \tilde{Y} \) be as in Section 2.3. Assume \( m \mid n = \text{codim}(S \hookrightarrow X) \) and \( n > m \). Then there is a semi-orthogonal decomposition
\[
\text{WC}(\varrho) = \langle i_*(\mathcal{E}), \Psi(\mathcal{D}^b_{\mu_m}(X)) \rangle
\]
where
\[
\mathcal{E} = \langle A(-m+1), A(-m+2), \ldots, A(-1), A \otimes \Omega^{n-m-1}(n-m-1), A \otimes \Omega^{n-m-2}(n-m-2), \ldots, A \otimes \Omega^n(m) \rangle.
\]
with \( A := \nu^*\mathcal{D}^b(S) \) and \( A(i) := A \otimes \Omega^i(i) \); the \( A \otimes \Omega^i(i) \) parts of the decomposition do not occur for \( n = 2m \). In particular, \( \text{WC}(\varrho) \) is an admissible subcategory of \( \mathcal{D}^b(\tilde{Y}) \).

**Proof.** We first want to show that \( \Psi(\mathcal{D}^b_{\mu_m}(X)) \subset \text{WC}(\varrho) \). For this, by Lemma 2.4.3, it is sufficient to show that \( \mathcal{L}^a_Y = \Psi(O_X \otimes \chi^a) \in \text{WC}(\varrho) \) for every \( a \in \{-m+1, \ldots, 0\} \).

The equivariant derived category \( \mathcal{D}^b_{\mu_m}(X) \) is a strongly (hence also weakly) crepant categorical resolution of the singularities of \( Y \) via the functors \( \pi^*, \pi_*^{\mu_m} \); see [Abu16, Thm. 1.0.2]. Since \( \Psi \circ \pi^* \cong g^* \) (see Lemma 2.4.4), \( C := \Psi(\mathcal{D}^b_{\mu_m}(X)) \) is a crepant resolution via the functors \( g^*, g_* \). Hence, \( g_*\mathcal{L}^a_Y \cong g_*\mathcal{L}^a_Y \) for \( a \in \{-m+1, \ldots, 0\} \) and it is only left to show that this isomorphism is induced by \( t \). Again by the \( Y \)-linearity of \( \Psi \), we have \( g_*\mathcal{L}^a_Y \cong \pi_* (O_X \otimes \chi^a)^{\mu_m} \) which is a reflexive sheaf on the normal variety \( Y \) (this follows for example by [Har80, Cor. 1.7]). By construction, \( t \) induces an isomorphism over \( Y \setminus S \). Since the codimension of \( S \) is at least 2, \( t : g_*\mathcal{L}^a_Y \to g_*\mathcal{L}^a_Y \) is an isomorphism of reflexive sheaves over all of \( Y \); see [Har80, Prop. 1.6].
By Theorem 2.4.1(ii), we have $D^b(\tilde{Y}) \cong \langle B, C \rangle$ with

$$B \cong i_*(\mathcal{A}(m-n), \ldots, \mathcal{A}(-1)) \cong i_*(\langle \mathcal{A}, \mathcal{A}(1), \ldots, \mathcal{A}(m-1) \rangle \perp).$$

We have $\rho_* B = 0$. It follows that $WC(\rho) = \langle B \cap \ker(\rho), C \rangle$. Indeed, consider an object $A \in D^b(\tilde{Y})$. It fits into an exact triangle $C \to A \to B \to$ with $C \in C$ and $B \in B$. From the morphism of triangles

$$g_*(C) \longrightarrow g_*(A) \longrightarrow g_*(B) = 0 \longrightarrow$$

we see that $t(A)$ is an isomorphism if and only if $g_! B = 0$.

It is left to compute $B \cap \ker(\rho_!)$. Let $F \in D^b(Z)$ and $B = i_* F$. By Corollary 2.4.14,

$$g_! B \cong g_! i_* F \cong b_* \nu_*(F \otimes \mathcal{O}_\nu(m-n)).$$

We see that $B \in \ker \rho_!$ if and only if $\nu_*(F \otimes \mathcal{O}_\nu(m-n)) = 0$ if and only if $F \in \nu^* D^b(S)(n-m)^\perp$.

Hence, $B \cap \ker \rho_! = i_*(F^\perp)$ with

$$\mathcal{F} = \langle \mathcal{A}, \mathcal{A}(1), \ldots, \mathcal{A}(m-1), \mathcal{A}(n-m) \rangle \subset D^b(Z)$$

Carrying out the appropriate mutations within the semi-orthogonal decomposition

$$D^b(Z) = \langle \mathcal{A}(-m+1), \mathcal{A}(-m+2), \ldots, \mathcal{A}(n-m-1), \mathcal{A}(n-m) \rangle,$$

we see that $\mathcal{F}^\perp = \mathcal{E}$; compare Lemma 2.2.3.

Since $\mathcal{E} \subset \langle \mathcal{A}(m-n), \ldots, \mathcal{A}(-1) \rangle$ is an admissible subcategory, we find that $i_* : \mathcal{E} \to D^b(\tilde{Y})$ is fully faithful and has adjoints. Hence, $WC(\rho) \subset D^b(\tilde{Y})$ is admissible. \hfill $\square$

**Remark 2.5.5.** We have $D^b(\tilde{Y}) = \langle \mathcal{A} \otimes \Omega^{n-m}(n-m), WC(\rho) \rangle$. In other words, we can achieve categorical weak crepancy by dropping only one $D^b(S)$ part of the semi-orthogonal decomposition of $D^b(\tilde{Y})$.

### 2.5.3 The discrepant category and some speculation

Let $Y$ be a variety with rational Gorenstein singularities and $\rho: \tilde{Y} \to Y$ a resolution of singularities. Then, $\rho$ is a crepant resolution if and only if $D^b(\tilde{Y}) = WC(\rho)$; compare [Abu16, Prop. 2.0.10]. We define the *discrepant category* of the resolution as the Verdier quotient

$$\text{disc}(\rho) := D^b(\tilde{Y})/WC(\rho).$$

By [Nee01, Remark 2.1.10], since $WC(\rho)$ is a kernel, and hence a thick subcategory, we have $\text{disc}(\rho) = 0$ if and only if $D^b(\tilde{Y}) = WC(\rho)$. Therefore, we can regard $\text{disc}(\rho)$ as a categorical measure of the discrepancy of the resolution $\rho: \tilde{Y} \to Y$.

In our cyclic quotient setup, where $\tilde{Y} \cong \text{Hilb}^G(X)$ is the simple blow-up resolution, we have $\text{disc}(\rho) \cong D^b(S)$ by Remark 2.5.5 and [LS16, Lem. A.8]. Hence, in this case, $\text{disc}(\rho)$ is the smallest non-zero category that one could expect (this is the most obvious in the case that $S$ is a point). This agrees with the intuition that the blow-up resolution is minimal in some way.
Question 2.5.6. Given a variety $Y$ with rational Gorenstein singularities, is there a resolution $\varrho: \tilde{Y} \to Y$ of minimal categorical discrepancy in the sense that, for every other resolution $\varrho': Y' \to Y$, there is a fully faithful embedding $\text{disc}(\varrho) \hookrightarrow \text{disc}(\varrho')$?

Often, in the case of a quotient singularity, a good candidate for a resolution of minimal categorical discrepancy should be the $G$-Hilbert scheme.

At least, we can see that $\text{disc}(\varrho)$ grows if we further blow-up the resolution.

Proposition 2.5.7. Let $\varrho: \tilde{Y} \to Y$ be a resolution of singularities and let $f: Y' \to \tilde{Y}$ be the blow-up in a smooth center $C \subset \tilde{Y}$. Set $\varrho': \varrho f: Y' \to Y$. Then there is a semi-orthogonal decomposition

$$\text{disc}(\varrho') = \langle R, \text{disc}(\varrho) \rangle \text{ with } R \neq 0.$$ 

Proof. We have a semi-orthogonal decomposition $D^b(Y') = \langle A, B \rangle$ with $B = f^*D^b(\tilde{Y})$ and

$$A = \langle t_*(g^*D^b(C) \otimes \mathcal{O}_g(-c + 1)), \ldots, t_*(g^*D^b(C) \otimes \mathcal{O}_g(-1)) \rangle.$$ 

Here, $c = \text{codim}(C \to \tilde{Y})$ and $g$ and $t$ are the $\mathbb{P}^{n-1}$-bundle projection and the inclusion of the exceptional divisor of the blow-up. We get a semi-orthogonal decomposition

$$\text{disc}(\varrho') = \langle A/(\text{WC}(\varrho') \cap A), B/(\text{WC}(\varrho') \cap B) \rangle;$$

this can be deduced from [LS16, Prop. B.2]. We have $f_*\mathcal{O}_{Y'} \cong \mathcal{O}_Y \cong f_*\omega_f$. By projection formula, it follows that $f_*\mathcal{O}_B \cong f_*\omega_f$. Hence, we have $B \cap \text{WC}(\varrho') = f^*\text{WC}(\varrho)$ and $B/(\text{WC}(\varrho') \cap B) \cong \text{disc}(\varrho)$. Thus, to get the assertion, it is sufficient to prove that $\text{WC}(\varrho') \cap A$ is a proper subcategory of $A$. Note that $f_*(A) = 0$, hence $g'_*(A) = 0$ and $\text{WC}(\varrho') \cap B = \ker(g'_*|A)$. Let $x \in C$ be a point and $E_x \subset \tilde{Y}$ the fibre of $f$ over $x$. We consider

$$A := \mathcal{O}_{E_x}(-c + 1) \in t_*(g^*D^b(C) \otimes \mathcal{O}_g(-c + 1)) \subset A.$$ 

We have $f_!A \cong f_*\mathcal{O}_{E_x} \cong \mathcal{O}_x$. Hence, $g'_!A \cong g'_*(\mathcal{O}_x) \neq 0$. \qed

Remark 2.5.8. One can show that the category $R$ is always a quotient of $D^b(C)$.

2.5.4 (Non-)unicity of categorical crepant resolutions

Let $\tilde{Y} \to Y$ be a resolution of rational Gorenstein singularities and let $D \subset D^\text{perf}(\tilde{Y})$ be an admissible subcategory which is a weakly crepant resolution, i.e. $\varrho^*: D^\text{perf}(\tilde{Y}) \to D^b(\tilde{Y})$ factors through $D$ and $\varrho^*: D \cong \varrho^*|D$. Then every admissible subcategory $D' \subset D$ with the property that $\varrho^*: D^\text{perf}(\tilde{Y}) \to D^b(\tilde{Y})$ factors through $D'$ is a weakly crepant resolution too.

In particular, in our setup of cyclic quotients, there is a tower of weakly crepant resolutions of length $n - m$ given by successively dropping the $D^b(S)$ parts of the semi-orthogonal decomposition of $\text{WC}(\varrho)$. We see that weakly crepant categorical resolutions are not unique, even if we fix the ambient derived category $D^b(\tilde{Y})$ of a geometric resolution $\tilde{Y} \to Y$.

In contrast, strongly crepant categorical resolutions are expected to be unique up to equivalence; see [Kuz08, Conj. 4.10]. A strongly crepant categorical resolution of $D^b(\tilde{Y})$ is a module category over $D^b(\tilde{Y})$ with trivial relative Serre functor; see [Kuz08, Sect. 3]. For an admissible subcategory $D \subset D^b(\tilde{Y})$ of the derived category of a geometric resolution of singularities $\varrho: \tilde{Y} \to Y$ this condition means that $D$ is $Y$-linear and there are functorial isomorphisms

$$g_*\text{Hom}(A, B)^\vee \cong g_*\text{Hom}(B, A) \quad (2.14)$$
for \(A, B \in \mathcal{D}\). In our cyclic setup, \(\Psi(D_G(X)) \subset \mathcal{D}^b(\tilde{Y})\) is a strongly crepant categorical resolution; see [Kuz08, Thm. 1] or [Abu16, Thm. 10.2].

We require strongly crepant categorical resolutions to be indecomposable which means that they do not decompose into direct sums of triangulated categories or, in other words, they do not admit both-sided orthogonal decompositions. Under this additional assumption, we can prove that strongly crepant categorical resolutions are unique if we fix the ambient derived category of a geometric resolution.

**Proposition 2.5.9.** Let \(\tilde{Y} \to Y\) be a resolution of Gorenstein singularities and \(\mathcal{D}, \mathcal{D}' \subset \mathcal{D}^b(\tilde{Y})\) admissible indecomposable strongly crepant subcategories. Then \(\mathcal{D} = \mathcal{D}'\).

**Proof.** The intersection \(\mathcal{D} \cap \mathcal{D}'\) is again an admissible \(Y\)-linear subcategory of \(\mathcal{D}^b(\tilde{Y})\) containing \(\varrho^*(\mathcal{D}^\text{perf}(Y))\). Furthermore, condition (2.14) is satisfied for every pair of objects of \(\mathcal{D} \cap \mathcal{D}'\) so that the intersection is again a strongly crepant resolution. Hence, we can assume without loss of generality that \(\mathcal{D}' \subset \mathcal{D}\).

Let \(A\) be the right-orthogonal complement of \(\mathcal{D}'\) in \(\mathcal{D}\) so that we have a semi-orthogonal decomposition \(\mathcal{D} = \langle A, \mathcal{D}' \rangle\). By Lemma 2.2.6, this means that \(\varrho_* \text{Hom}(D, A) = 0\) for \(A \in A\) and \(D \in \mathcal{D}'\). But then, by (2.14), we also get \(\varrho_* \text{Hom}(A, D) = 0\) so that \(\mathcal{D} = A \oplus \mathcal{D}'\). \(\square\)

### 2.5.5 Connection to Calabi-Yau neighbourhoods

In [HKP16], spherelike objects and spherical subcategories generated by them were introduced and studied. The paper gave some evidence that these objects and subcategories might play a role in birationality questions for Calabi-Yau varieties. One of the starting points for our project was the idea to consider spherical subcategories, and their generalisations Calabi-Yau neighbourhoods, as candidates for categorical crepant resolutions of Calabi-Yau quotient varieties. In this subsection, we briefly describe the connection to the weakly crepant resolutions considered above.

We recall some abstract homological notions. Let \(\mathcal{T}\) be a Hom-finite \(\mathbb{C}\)-linear triangulated category and \(E \in \mathcal{T}\) an object. We say that \(SE \in \mathcal{T}\) is a Serre dual object for \(E\) if the functors \(\text{Hom}^*(E, -)\) and \(\text{Hom}^*(-, SE)^\vee\) are isomorphic. By the Yoneda lemma, \(SE\) is then uniquely determined. Fix an integer \(d\). We call the object \(E\)

- a \(d\)-Calabi-Yau object, if \(E[d]\) is a Serre dual of \(E\),
- \(d\)-spherelike if \(\text{Hom}^*(E, E) = \mathbb{C} \oplus \mathbb{C}[-d]\), and
- \(d\)-spherical if \(E\) is \(d\)-spherelike and a \(d\)-Calabi-Yau object.

Note a smooth compact variety \(X\) of dimension \(d\) is a strict Calabi-Yau variety precisely if the structure sheaf \(\mathcal{O}_X\) is a \(d\)-spherical object of \(\mathcal{D}^b(X)\).

In [HKP16] the authors show that if \(E\) is a \(d\)-spherelike object, there exists a unique maximal triangulated subcategory of \(\mathcal{T}\) in which \(E\) becomes \(d\)-spherical. In the following we will imitate this construction for a larger class of objects.

**Definition 2.5.10.** Let \(E \in \mathcal{T}\) be an object in a triangulated category having a Serre dual \(SE\). We call \(E\) a \(d\)-selfdual object if

1. \(\text{Hom}(E, E[d]) \cong \mathbb{C}\), i.e. by Serre duality there is a morphism \(w : E \to \omega(E) := SE[-d]\) unique up to scalars, and
2. the induced map $w_* : \text{Hom}^*(E, E) \xrightarrow{\sim} \text{Hom}^*(E, \omega(E))$ is an isomorphism.

In particular, a $d$-selfdual object satisfies $\text{Hom}^*(E, E) \cong \text{Hom}(E, E)^\vee[-d]$, hence the name.

**Remark 2.5.11.** If an object is $d$-spherelike, then it is $d$-selfdual; compare [HKP16, Lem. 4.2].

For a $d$-selfdual object $E$, there is a triangle $E \to \omega(E) \to Q_E \to E[1]$ induced by $w$. By our assumption, we get $\text{Hom}^*(E, Q_E) = 0$. Thus, following an idea suggested by Martin Kalck after discussing [HKP16, Å§7] with Michael Wemyss, we propose the following

**Definition 2.5.12.** The Calabi-Yau neighbourhood of a $d$-selfdual object $E \in \mathcal{T}$ is the full triangulated subcategory

$$\text{CY}(E) := \downarrow Q_E \subseteq \mathcal{T}.$$ 

**Proposition 2.5.13.** If $E \in \mathcal{T}$ is a $d$-selfdual object then $E \in \text{CY}(E)$ is a $d$-Calabi-Yau object.

**Proof.** If $T \in \text{CY}(E)$, apply $\text{Hom}^*(T, -)$ to the triangle $E \to \omega(E) \to Q_E$. 

Using the same proof as for [HKP16, Thm. 4.6], we see that the Calabi-Yau neighbourhood is the maximal subcategory of $\mathcal{T}$ in which a $d$-selfdual object $E$ becomes $d$-Calabi-Yau.

**Proposition 2.5.14.** If $\mathcal{U} \subset \mathcal{T}$ is a full triangulated subcategory and $E \in \mathcal{U}$ is $d$-Calabi-Yau, then $\mathcal{U} \subset \text{CY}(E)$.

**Proposition 2.5.15.** Let $Y$ be a projective variety with rational Gorenstein singularities and trivial canonical bundle of dimension $d = \dim Y$ and consider a resolution of singularities $\varrho : \tilde{Y} \to Y$. Then, for every line bundle $L \in \text{Pic} Y$, the pull-back $\varrho^* L \in \text{D}^b(\tilde{Y})$ is $d$-selfdual. Furthermore, we have

$$\text{WC} (\varrho) = \bigcap_{L \in \text{Pic} Y} \text{CY}(\varrho^* L).$$

(2.15) 

**Proof.** Note that, by our assumption that $\omega_Y$ is trivial, we have $\omega_{\tilde{Y}} \cong \omega_{\varrho}$. Hence, by Grothendieck duality, there is a morphism $w_L : \varrho^* L \to \varrho^* L \otimes \omega_{\tilde{Y}}$ unique up to scalar multiplication, namely $w_L = \text{id}_{\varrho^* L} \otimes s$ where $s$ is the non-zero section of $\omega_{\tilde{Y}} \cong \omega_{\varrho}$; compare the previous Subsection 2.5.1. Furthermore, $w_L^* : \text{Hom}^*(\varrho^* L, \varrho^* L) \to \text{Hom}^*(\varrho^* L, \varrho^* L \otimes \omega_{\tilde{Y}})$ is an isomorphism, still by Grothendieck duality, which means that $\varrho^* L$ is $d$-selfdual.

Recall that $\text{WC}(\varrho) = \ker(\varrho_+)$ where $\varrho_+$ is defined as the cone

$$\varrho_+ \xrightarrow{\iota} \varrho \to \varrho_+ \to .$$

By adjunction, we get $\text{WC}(\varrho) = \downarrow (\varrho_+ (\text{D}^{\text{perf}} (Y)))$ where $\varrho_+ = \varrho_+^R$ is given by the triangle

$$\varrho_+ \to \varrho^* \xrightarrow{t^R} \varrho^* \to .$$

Note that $t^R = (\_ ) \otimes s$. Hence $t^R (L) = w_L : \varrho^* L \to \varrho^* L \otimes \omega_{\tilde{Y}}$ and $\varrho^+ (L) = Q_{\varrho^* L}[1]$; compare Definition 2.5.12. Since the line bundles form a generator of $\text{D}^{\text{perf}} (Y)$, we get for $F \in \text{D}^b(\tilde{Y})$:

$$F \in \text{WC}(\varrho) \iff F \in \downarrow (\varrho_+ (\text{D}^{\text{perf}} (Y))) \iff F \in \downarrow Q_{\varrho^* L} \quad \forall L \in \text{Pic} Y \iff F \in \text{CY}(\varrho^* L) \quad \forall L \in \text{Pic} Y .$$
Using [CP10, Prop. 3.5(b)], this then implies that \langle A \rangle in singular Kummer variety of object. The same should hold in general if \( Y \) has weakly crepant neighbourhood is computed by a Calabi-Yau neighbourhood of a single object. The same hold in general if \( Y \) has isolated singularities.

### 2.6 Stability conditions for Kummer threefolds

Let \( A \) be an abelian variety of dimension \( g \). Consider the action of \( G = \mu_2 \) by \( \pm 1 \). Then the fixed point set \( A[2] \) consists of the 4\(^2\) two-torsion points. Consider the quotient \( \overline{A} \) (the singular Kummer variety) of \( A \) by \( G \), and the blow-up \( K(A) \) (the Kummer resolution) of \( \overline{A} \) in \( A[2] \). This setup satisfies Condition 2.3.1, with \( m = 2 \) and \( n = g \) and we get

**Corollary 2.6.1.** The functor \( \Psi : D^b_G(A) \to D^b(K(A)) \) is fully faithful, and

\[
D^b(K(A)) = \langle \underbrace{D^b(pt), \ldots, D^b(pt)}_{(g-2)^4 \text{ times}}, \Psi(D^b_G(A)) \rangle.
\]

To explore a potentially useful consequence of this result, we need to recall that a Bridgeland stability condition on a reasonable \( \mathbb{C} \)-linear triangulated category \( \mathcal{D} \) consists of the heart \( \mathcal{A} \) of a bounded t-structure in \( \mathcal{D} \) and a function from the numerical Grothendieck group of \( \mathcal{D} \) to the complex numbers satisfying some axioms, see [Bri07].

**Corollary 2.6.2.** There exists a Bridgeland stability condition on \( D^b(K(A)) \), for an abelian threefold \( A \).

**Proof.** To begin with, one checks that \( D^b_G(A) \) has a stability condition, which follows quite easily from [BMS16, Cor. 10.3]. For a two-torsion point \( x \in A[2] \), we set \( E_x := O_{\hat{g}^{-1}(\pi(x))}(-1) \). Then, since \( g = \dim A = 3 \), the semi-orthogonal decomposition of Corollary 2.6.1 is given by

\[
D^b(K(A)) = \langle \{ E_x \}_{x \in A[2]}, D^b_G(A) \rangle.
\]

Next, we want to show that, for every \( x \in A[2] \), there exists an integer \( i \) such that \( \text{Hom}^{\leq i}(E_x, \Psi(F)) = 0 \) for all \( F \in \mathcal{A} \subset D^b_G(A) \). Indeed, the cohomology of any complex in the heart of the stability condition on \( D^b_G(A) \), as constructed in [BMS16, Cor. 10.3], is concentrated in an interval of length three. The functor \( \Psi \) has cohomological amplitude at most 3, since \( q^G : \text{Coh}_G(A) \to \text{Coh}(K(A)) \) is an exact functor of abelian categories, and every sheaf on \( A \) has a locally free resolution of length \( \dim A = 3 \). This implies that the cohomology of any complex in \( \Psi(D^b_G(A)) \) is contained in a fixed interval of length 6. This proves the above claim. Using [CP10, Prop. 3.5(b)], this then implies that \( \langle E_x, \Psi(D^b_G(A)) \rangle \) has a stability condition; compare the argument in [Ber+12, Cor. 3.8].

We can proceed to show that, for \( x \neq y \in A[2] \), there exists an integer \( i \) such that \( \text{Hom}^{\leq i}(E_y, \langle E_x, \Psi(D^b_G(A)) \rangle) = 0 \) and hence there is a stability condition on \( \langle E_y, E_x, \Psi(D^b_G(A)) \rangle \). After \( 4^3 \) steps we have constructed a stability condition on \( D^b(K(A)) \); compare (2.16). \qed
References


\[ G = \mu_m = \langle g \rangle \text{ acts on smooth } X \]
\[ S = \text{Fix}(G) \subset X, n = \dim(X) - \dim(S) \]
\[ N = N_{S/X} \text{ with } g|_N = \zeta \cdot \text{id}_N \]
\[ \chi: G \to \mathbb{C}^*, \chi(g) = \zeta^{-1} \]
\[ \mathcal{L}\mathcal{Y} \in \text{Pic}(\mathcal{Y}) \text{ with } \mathcal{L}\mathcal{Y}^m = \mathcal{O}\mathcal{Y}(Z) \]
\[ \mathcal{L}\mathcal{X} = \mathcal{O}\mathcal{X}(Z) \in \text{Pic}^G(\mathcal{X}) \text{ with trivial action on } \mathcal{L}\mathcal{X}|_Z = \mathcal{O}\mathcal{X}(Z) = \mathcal{O}\nu(-1) \]
\[ \Phi := p_* \circ q^* \circ \text{triv} \]
\[ \Psi := (-)^G \circ q_* \circ p^* \]
\[ \Theta_\beta := i_* (\nu^*(\_ \otimes \mathcal{O}_\nu(\beta))) \]
\[ \Xi_\alpha := (a_* \circ \text{triv}) \otimes \chi^\alpha \]
Chapter 3

Spherical functors on the Kummer surface


Abstract

We find two natural spherical functors associated to the Kummer surface and analyse how their induced twists fit with Bridgeland’s conjecture on the derived autoequivalence group of a complex algebraic K3 surface.

3.1 Introduction

Let $\mathcal{D}(X)$ be the bounded derived category of coherent sheaves on a smooth complex projective variety $X$ and $\text{Aut}(\mathcal{D}(X))$ denote the set of isomorphism classes of exact $\mathbb{C}$-linear autoequivalences of $\mathcal{D}(X)$. Then we always have a subgroup $\text{Aut}_\text{st}(\mathcal{D}(X)) \subset \text{Aut}(\mathcal{D}(X))$ of *standard* autoequivalences which is generated by push forwards along automorphisms, twists by line bundles and shifts. The complement of this subgroup, if non-empty, is usually very interesting and mysterious; its elements will be called *non-standard* autoequivalences.

The most successful way to construct non-standard autoequivalences was discovered in the groundbreaking work of Seidel and Thomas [ST01] on *spherical objects*. This was extended by Huybrechts and Thomas [HT06] to a notion of $\mathbb{P}$-*objects* and further still, to a theory of *spherical* and $\mathbb{P}$-*functors*; see [Rou06; Ann08; Add16].

The first example of a series of $\mathbb{P}$-functors was constructed by Addington in [Add16, Theorem 2] for the Hilbert scheme $X^{[n]}$ of $n$ points on a K3 surface $X$. In particular, he showed that the natural functor $F : \mathcal{D}(X) \to \mathcal{D}(X^{[n]})$ induced by the universal ideal sheaf on $X \times X^{[n]}$ is a $\mathbb{P}^{n-1}$-functor in the sense of [Add16, §3] and thus gives rise to a non-standard autoequivalence of $\mathcal{D}(X^{[n]})$ for each $n \geq 2$. Notice that when $n = 1$, this $F$ is Mukai’s reflection functor [Muk87, p.362] which coincides (up to a shift) with the spherical twist around the structure sheaf $\mathcal{O}_X$.

Inspired by this example, the second author [Mea15, Theorem 4.1] provided the analogous result for the generalised Kummer variety $K_n \subset A^{[n+1]}$ associated to an abelian surface $A$. More precisely, he proved that the natural Fourier-Mukai functor $F_K : \mathcal{D}(A) \to \mathcal{D}(K_n)$ induced
by the universal ideal sheaf on \( A \times K_n \) is again a \( \mathbb{P}^{n-1} \)-functor yielding a new non-standard autoequivalence of \( \mathcal{D}(K_n) \) for each \( n \geq 2 \).

This short note completes this theorem to the case \( n = 1 \) where the generalised Kummer variety is the classical Kummer surface. The motivation to understand this particular case comes from Bridgeland’s conjecture [Bri08, Conjecture 1.2] on the derived autoequivalence group of a complex algebraic K3 surface; roughly speaking, it says that \( \text{Aut}(\mathcal{D}(X)) \) should be generated by standard autoequivalences and twists around spherical objects.

**Summary of main results**

Every abelian surface \( A \) has a natural K3 surface associated to it; namely the Kummer surface \( K := K_1 \). It can either be defined as the blow up of the quotient \( A/\iota \) along the sixteen ordinary double points, where \( \iota \) denotes the involution \( a \mapsto -a \), or equivalently as the fibre of the Albanese map \( m : A^{[2]} \to A \) over zero. That is, we can identify \( K \) with the subvariety of the Hilbert scheme \( A^{[2]} \) consisting of those points representing length 2 subschemes of \( A \) whose weighted support sums to zero. In other words, there is a universal family \( Z \subset A \times K \) giving rise to the commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{q} & K \\
\downarrow{p} & & \downarrow{\mu} \\
A & \xrightarrow{\pi} & A/\iota
\end{array}
\]

Recall that a Fourier-Mukai functor \( F : \mathcal{D}(Y) \to \mathcal{D}(X) \) with left adjoint \( L \) and right adjoint \( R \) is said to be spherical if the cotwist \( C_F := \text{cone}(\text{id}_Y \to RF) \) is an autoequivalence of \( \mathcal{D}(Y) \) and we have a functorial isomorphism \( R \cong CL \). In particular, if \( F \) is spherical then the twist \( T_F := \text{cone}(FR \to \text{id}) \) is an autoequivalence of \( \mathcal{D}(X) \). A spherical object \( \mathcal{E} \in \mathcal{D}(X) \) corresponds to the case \( F := (\_ \otimes \mathcal{E}) : \mathcal{D}(\text{pt}) \to \mathcal{D}(X) \).

In this article, we focus on the exact triangle \( F \to F' \to F'' \) of Fourier-Mukai functors \( \Phi_E : \mathcal{D}(A) \to \mathcal{D}(K) \) induced by the structure sequence of \( Z \):

\[
F := \Phi_{Z} \quad F' := \Phi_{O_{A \times K}} = H^*(\_ \otimes O_K) \quad F'' := \Phi_{Z} = q_\ast p^*.
\]

Our main result is the following

**Theorem (3.2.1 and 3.2.4).** Both \( F \) and \( F'' \) are spherical functors with cotwists \( C_F \simeq C_{F''} \simeq \iota^* \).

In light of [Bri08, Conjecture 1.2], this immediately raises the question whether the twists \( T_F, T_{F''} \in \text{Aut}(\mathcal{D}(K)) \) associated to these functors \( F, F'' \) can be decomposed into twists \( T_E \) around spherical objects \( \mathcal{E} \in \mathcal{D}(K) \). We answer this question with the following

**Theorem (3.2.1 and 3.2.4).** The induced twists \( T_F, T_{F''} \in \text{Aut}(\mathcal{D}(K)) \) decompose in the following way:

\[
T_{F''} \simeq \prod_i T_{\mathcal{O}_{E_i}(-1)} \circ M_{\mathcal{O}_K(E/2)[1]} \simeq \prod_i T_{\mathcal{O}_E} \circ M_{\mathcal{O}_K(-E/2)[1]}
\]
and
\[ F[1] \simeq T_{O_K} \circ F'' \implies T_F \simeq T_{O_K} \circ T_{F''} \circ T_{O_K}^{-1} \]
where \( E = \bigcup_i E_i \) for the exceptional curves \( E_i \) of the Hilbert-Chow morphism \( \mu \) and \( M_{O_K(E/2)} := (\_ \otimes O_K(E/2)) \).

It is easy to see that the squares \( T_F^2, T_{F''}^2 \) of our twists act trivially on the cohomology of \( K \) (see [Add16, §1.4]). In fact, Corollary 3.2.5 shows that \( T_F^2 \simeq T_{F''}^2 \simeq [2] \).

In this paper, we will give a different proof of Theorem 3.2.4 to that which could have been obtained from adapting the arguments in [Mea15]. The advantage of our approach is that it immediately provides us with the decompositions of \( T_F \) and \( T_{F''} \) as stated above.

Acknowledgements: We thank Nick Addington and Will Donovan for helpful discussions as well as the Hausdorff Research Institute for Mathematics (HIM) for their excellent hospitality whilst this work was carried out. C.M. is very grateful to Arend Bayer for his consistent help and support. A.K. was supported by the SFB/TR 45 ‘Periods, Moduli Spaces and Arithmetic of Algebraic Varieties’ of the DFG (German Research Foundation) and C.M. was supported by an EPSRC Doctoral Prize Research Fellowship Grant no. EP/K503034/1.

3.2 Natural Functors on the Kummer Surface

Another way of describing \( K \) is by first blowing-up the fixed points \( \tilde{A} \to A \). Since the fixed points are \( \iota \)-invariant, the involution \( \iota \) lifts to an involution \( \tilde{\iota} \) of \( \tilde{A} \).

\[ \begin{array}{ccc}
\tilde{A} & \xrightarrow{p} & A \\
\downarrow q & & \downarrow \pi \\
K & \xrightarrow{\mu} & A/\iota
\end{array} \]

The quotient \( \tilde{A} \to K \) is a double cover ramified over sixteen exceptional curves \( E_i \). Moreover, the canonical bundle formula for the blow-up yields \( \omega_{\tilde{A}} \simeq \mathcal{O}(\sum \tilde{E}_i) \) where the \( \tilde{E}_i \) are the exceptional divisors in \( \tilde{A} \). Their images \( E_i \) in \( K \) satisfy \( q^* \mathcal{O}(E_i) \simeq \mathcal{O}(2\tilde{E}_i) \) and \( q_* \mathcal{O}_{\tilde{A}} \simeq \mathcal{O}_K \oplus \mathcal{O}(-\frac{1}{2} \sum E_i) \). See [Huy16, Chapter 1.1] for more details. We set \( E := \bigcup_i E_i \) and \( \tilde{E} := \bigcup_i \tilde{E}_i \) from now on.

**Proposition 3.2.1.** \( F' : \mathcal{D}(A) \to \mathcal{D}(K) \) is a spherical functor with cotwist \( C_{F''} \simeq \iota^* \) and twist
\[ T_{F''} \simeq \prod_i T_{O_{E_i}(-1)}^{-1} \circ M_{O_K(E/2)}[1]. \]

**Proof.** Pushforward along the double cover \( q_* : \mathcal{D}(\tilde{A}) \to \mathcal{D}(K) \) is a spherical functor with cotwist \( C_{q_*} \simeq M_{O_{\tilde{A}}(E)} \circ \tilde{\iota}^* \simeq S_{\tilde{A}} \circ \tilde{\iota}^*(-2) \) and twist \( T_{q_*} \simeq M_{O_K(E/2)}[1] \); see [Add16, §1.2, Examples 5 & 6].
By [Orl92, Theorem 4.3], we have a semi-orthogonal decomposition

$$\mathcal{D}(\mathbb{A}) \simeq \langle \mathcal{O}_{E_1}(-1), \ldots, \mathcal{O}_{E_{16}}(-1), p^* \mathcal{D}(A) \rangle$$

We set $\mathcal{A} := \langle \mathcal{O}_{E_1}(-1), \ldots, \mathcal{O}_{E_{16}}(-1) \rangle$ and $\mathcal{B} := p^* \mathcal{D}(A)$ so that $\mathcal{D}(\mathbb{A}) \simeq \langle \mathcal{A}, \mathcal{B} \rangle$. Since $\mathcal{D}(\mathbb{A}) \simeq \langle S_A \mathcal{B}, \mathcal{A} \rangle$ by [BK89] and $C_B \mathcal{B} \simeq S_A \mathcal{B}$, we have $\mathcal{D}(\mathbb{A}) \simeq \langle C_B \mathcal{B}, \mathcal{A} \rangle$. Thus, by [HS16, Theorem 4.13], the restrictions $q_\ast |_A : \mathcal{D}(A[2]) \to \mathcal{D}(K)$ (to the set $A[2] \subset A$ of 2-torsion points) and $q_\ast |_B \simeq q_\ast p^* = : F'' : \mathcal{D}(A) \to \mathcal{D}(K)$ are spherical functors with $T_{q_\ast} \simeq T_{q_\ast |_A} \circ T_{q_\ast |_B}$.

Since $q_\ast \mathcal{O}_{E_1}(-1) \simeq \mathcal{O}_{E_1}(-1)$, we see that $T_{q_\ast |_A} \simeq \prod_i T_{\mathcal{O}_{E_i}(-1)}$ and hence

$$T_{F''} \simeq T_{q_\ast |_A} \circ T_{q_\ast} \simeq \prod_i T_{\mathcal{O}_{E_i}(-1)} \circ M_{\mathcal{O}_K(E/2)}[1].$$

Notice that the cotwist of $F'' \simeq q_\ast |_B$ is given by $S_A \circ \iota^*[2] \simeq \iota^*$. \hfill $\square$

**Remark 3.2.2.** We can use equation (3.1) below to rewrite this decomposition as

$$T_{F''} \simeq \prod_i T_{\mathcal{O}_{E_i}} \circ M_{\mathcal{O}_K(-E/2)}[1].$$

**Lemma 3.2.3.** We have the following isomorphism of functors

$$F[1] \simeq T_{\mathcal{O}_K} \circ F''.$$  

**Proof.** Consider the following exact triangles of functors

$$\text{Hom}^*(\mathcal{O}_K, F'') \otimes \mathcal{O}_K \to F'' \to T_{\mathcal{O}_K} \circ F'' \quad \text{and} \quad F' \to F'' \to F[1].$$

Then it is sufficient to show that $\text{Hom}^*(\mathcal{O}_K, F'') \otimes \mathcal{O}_K \simeq F' \simeq H^*(A, \_ \mathcal{O}_K).$ In other words, it is enough to show that $H^*(K, F''(\_)) \simeq H^*(A, \_)$ but this follows from the fact that $p$ is a blowup. Indeed, we have

$$H^*(K, F''(\_)) \simeq H^*(K, q_\ast p^*(\_)) \simeq H^*(\mathbb{A}, p^*(\_)) \simeq H^*(A, p_\ast p^*(\_)) \simeq H^*(A, \_).$$ \hfill $\square$

**Corollary 3.2.4.** $F : \mathcal{D}(A) \to \mathcal{D}(K)$ is a spherical functor with cotwist $C_F \simeq \iota^*$ and twist

$$T_F \simeq T_{\mathcal{O}_K} \circ T_{F''} \circ T_{\mathcal{O}_K}^{-1}.$$  

**Proof.** Recall that if $F : \mathcal{D}(Z) \to \mathcal{D}(Y)$ is a spherical functor and $\Phi : \mathcal{D}(Y) \overset{\sim}{\to} \mathcal{D}(X)$ is an equivalence of categories then $\Phi \circ F : \mathcal{D}(Z) \to \mathcal{D}(X)$ is also a spherical functor with the same cotwist and $T_{\Phi \circ F} \simeq \Phi \circ T_F \circ \Phi^{-1}$. In particular, we see immediately from Lemma 3.2.3 that $F$ is a spherical functor with cotwist $C_F \simeq \iota^*$ and twist

$$T_F \simeq T_{F[1]} \simeq T_{\mathcal{O}_K} \circ T_{F''} \circ T_{\mathcal{O}_K}^{-1}.$$ \hfill $\square$

**Corollary 3.2.5.** The squares of the spherical twists are given by

$$T_F^2 \simeq T_{F''}^2 \simeq [2].$$

In particular, $T_F^2, T_{F''}^2$ act trivially on cohomology.
Then, by [Orl92, Theorem 2.6], we have a semi-orthogonal decomposition

\[ \mathcal{D}(E) \simeq \langle A_1, A_2 \rangle \]

Thus, using Kuznetsov’s trick [AA13, Theorem 11] (which is a special case of [HS16, Theorem 4.13]), we see that the restriction \( j_\ell := j_\ell |_{A_\ell} : \mathcal{D}(A[2]) \to \mathcal{D}(K) \) is spherical for each \( \ell = 1, 2 \) and the twists satisfy \( T_{j_\ell} \circ T_{j_\ell} \simeq T_{j_\ell} \). That is

\[ \prod_i T_{\mathcal{O}_{E_i}(-1)} \cdot \prod_i T_{\mathcal{O}_{E_i}} \simeq M_{\mathcal{O}_K(E)}. \]  

(3.1)

Furthermore, we have \( j_1 \simeq M_{\mathcal{O}_K(E/2)} \circ j_2 \) since \( \mathcal{O}_{E_i}(E/2) \simeq \mathcal{O}_{E_i}(-1) \) and so

\[ T_{j_1} \simeq T_{M_{\mathcal{O}_K(E/2)} \circ j_2} \simeq M_{\mathcal{O}_K(E/2)} \circ T_{j_2} \circ M_{\mathcal{O}_K(-E/2)} \]

which, after taking inverses, equates to

\[ \prod_i T_{\mathcal{O}_{E_i}(-1)}^{-1} \circ M_{\mathcal{O}_K(E/2)} \simeq M_{\mathcal{O}_K(E/2)} \circ \prod_i T_{\mathcal{O}_{E_i}}^{-1}. \]  

(3.2)

This expression allows us to reduce the formula for \( T^2_F \) in the following way:

\[ T^2_F \simeq \prod_i T_{\mathcal{O}_{E_i}(-1)} \circ M_{\mathcal{O}_K(E/2)} \circ \prod_i T_{\mathcal{O}_{E_i}(-1)}^{-1} \circ M_{\mathcal{O}_K(E/2)}[2] \]

\[ \simeq M_{\mathcal{O}_K(E/2)} \circ \prod_i T_{\mathcal{O}_{E_i}(-1)}^{-1} \circ \prod_i T_{\mathcal{O}_{E_i}(-1)}^{-1} \circ M_{\mathcal{O}_K(E/2)}[2] \]

\[ \simeq M_{\mathcal{O}_K(E/2)} \circ M_{\mathcal{O}_K(-E)} \circ M_{\mathcal{O}_K(E/2)}[2] \]

\[ \simeq [2] \]

where the second and third lines follow immediately from equations (3.2) and (3.1) respectively.

The fact that \( T^2_F \simeq [2] \) now follows immediately from Corollary 3.2.4.

**Corollary 3.2.6.** im \( F \) and im \( F'' \) are spanning classes for \( \mathcal{D}(K) \).

**Proof.** For any spherical functor \( F : \mathcal{D}(Y) \to \mathcal{D}(X) \), we have a natural spanning class for \( \mathcal{D}(X) \) given by im \( F \cup (im \, F^\perp) \simeq im \, F \cup ker \, R \); see [Add16, §1.4]. However, in our case we have ker \( R = 0 \). Indeed, let \( \mathcal{E} \in ker \, R \). Then the defining triangle for the twist \( FR(\mathcal{E}) \to \mathcal{E} \to TR(\mathcal{E}) \) shows that \( T_F(\mathcal{E}) \simeq \mathcal{E} \). But by Corollary 3.2.5, we have \( \mathcal{E} \simeq T^2_F(\mathcal{E}) \simeq [2] \) which implies \( \mathcal{E} \simeq 0 \); a similar argument works for \( F'' \). \( \square \)

**Remark 3.2.7.** This should be contrasted to the object case where every spherical object \( \mathcal{E} \) is expected to have a non-empty perpendicular \( \mathcal{E}^\perp \); [Plo07, Question 1.25].

**Lemma 3.2.8.** The functors \( F, F'' : \mathcal{D}(A) \to \mathcal{D}(K) \) are actually split spherical. That is, the natural triangles associated to the units \( \eta, \eta'' \) of adjunction are split. In particular, this implies that \( F \) and \( F'' \) are faithful.
Proof. We prove the statement only for \( F \) since \( F'' \) is identical. In order to show that the triangle \( \text{id}_A \xrightarrow{\eta} RF \rightarrow \iota^* \) is split, it suffices to show that \( \text{Ext}^1(\text{id}_A, \iota^*) = 0 \). But on the level of kernels, this is just

\[
\text{Ext}^1_{A \times A}(\Delta_*O_A, \Omega_F) \simeq \text{Ext}^1_A(O_A, \Delta^*\Omega_F) \text{ by adjunction}
\]

\[
\simeq \text{Ext}^1_A(O_A, \Delta^*\Omega_F[-2])
\]

\[
\simeq H^{-1}(A, O_A[2]) = 0.
\]

**Proposition 3.2.9.** The induced map on cohomology \( F^H : H^*(A, Q) \to H^*(K, Q) \) is injective on \( H^{\text{even}}(A, Q) \), zero on \( H^{\text{odd}}(A, Q) \) and the twist \( T_F \) acts on \( H^*(K, Q) \) by reflection in \( (\text{im} F^H)^\perp \) with respect to the Mukai pairing.

Proof. The first statement follows from the fact that \( R^H F^H \simeq \text{id}_{H^*(A, Q)} + \iota^H \) and \( \iota^H \) acts by the identity on \( H^{\text{even}}(A, Q) \) and by \( -1 \) on \( H^{\text{odd}}(A, Q) \). Next, the defining triangle for the twist gives \( T_F^H \simeq \text{id}_{H^*(K, Q)} - F^H R^H \) from which it follows immediately that everything in \( \ker R^H \simeq (\text{im} F^H)^\perp \) is fixed by \( T_F^H \). Finally, to see that \( T_F^H \) acts on \( \text{im} F^H \) as \( -1 \) we observe that \( T_F \circ F \simeq F \circ C_F[1] \simeq F \circ \iota^*[1] \simeq F[1] \) and so the claim follows.

**Remark 3.2.10.** Notice that this is very different to the object case where the twist acts on cohomology by reflection in a hyperplane; see [Huy06, Corollary 8.13] for more details. It follows from Proposition 3.2.9 that our twist is acting on cohomology by reflection in a subspace of codimension \( 8 = \dim H^{\text{even}}(A, Q) \).

**References**


Chapter 4

Varieties with \( \mathbb{P} \)-units


Abstract

We study the class of compact Kähler manifolds with trivial canonical bundle and the property that the cohomology of the trivial line bundle is generated by one element. If the square of the generator is zero, we get the class of strict Calabi–Yau manifolds. If the generator is of degree 2, we get the class of compact hyperkähler manifolds. We provide some examples and structure results for the cases where the generator is of higher nilpotency index and degree. In particular, we show that varieties of this type are closely related to higher-dimensional Enriques varieties.

4.1 Introduction

In this paper we will study a certain class of compact Kähler manifolds with trivial canonical bundle which contains all strict Calabi–Yau varieties as well as all hyperkähler manifolds. For the bigger class of manifolds with trivial first Chern class \( c_1(X) = 0 \in H^2(X, \mathbb{R}) \) there exists the following nice structure theorem, known as the \textit{Beauville–Bogomolov decomposition}; see [Bea83]. Namely, each such manifold \( X \) admits an étale covering \( X' \to X \) which decomposes as

\[
X' = T \times \prod_i Y_i \times \prod_j Z_j
\]

where \( T \) is a complex torus, the \( Y_i \) are hyperkähler, and the \( Z_j \) are simply connected strict Calabi–Yau varieties of dimension at least 3.

Given a variety \( X \), the graded algebra \( H^*(\mathcal{O}_X) := \bigoplus_{i=0}^{\dim X} H^i(X, \mathcal{O}_X)[-i] \) is considered an important invariant; see, in particular, Abuaf [Abu15] who calls \( H^*(\mathcal{O}_X) \) the \textit{homological unit} of \( X \) and conjectures that it is stable under derived equivalences. In this paper, we want to study varieties which have trivial canonical bundle and the property that the algebra \( H^*(\mathcal{O}_X) \) is generated by one element.

The main motivation are the following two observations. Let \( X \) be a compact Kähler manifold.
Observation 4.1.1. $X$ is a strict Calabi–Yau manifold if and only if the canonical bundle $\omega_X$ is trivial and $H^i(O_X) \cong \mathbb{C}[x]/x^2$ with $\deg x = \dim X$. These conditions can be summarised in terms of objects of the bounded derived category $D(X) := D^b(Coh(X))$ of coherent sheaves. Namely, $X$ is a strict Calabi–Yau manifold if and only if $O_X \in D(X)$ is a spherical object in the sense of Seidel and Thomas [ST01].

The above is a very simple reformulation of the standard definition of a strict Calabi–Yau manifold as a compact Kähler manifold with trivial canonical bundle such that $H^i(O_X) = 0$ for $i \neq 0, \dim X$. The second observation is probably less well-known.

Observation 4.1.2. $X$ is a hyperkähler manifold of dimension $\dim X = 2n$ if and only if $\omega_X$ is trivial and $H^*(O_X) \cong \mathbb{C}[x]/x^{n+1}$ with $\deg x = 2$. This is equivalent to the condition that $O_X \in D(X)$ is a $\mathbb{P}^n$-object in the sense of Huybrechts and Thomas [HT06].

Indeed, the structure sheaf of a hyperkähler manifold is one of the well-known examples of a $\mathbb{P}^n$-object; see [HT06, Ex. 1.3(ii)]. The fact that $H^*(O_X)$ also characterises the compact hyperkähler manifolds follows from [HN11, Prop. A.1].

Inspired by this, we study the class of compact Kähler manifolds $X$ with the property that $O_X \in D(X)$ is what we call a $\mathbb{P}^n[k]$-object; see Definition 4.2.4. Concretely, this means:

(C1) The canonical bundle $\omega_X$ is trivial,

(C2) There is an isomorphism of $\mathbb{C}$-algebras $H^*(O_X) \cong \mathbb{C}[x]/x^{n+1}$ with $\deg x = k$.

By Serre duality, such a manifold is of dimension $\dim X = \deg(x^n) = n \cdot k$. For $n = 1$, we get exactly the strict Calabi–Yau manifolds while for $k = 2$ we get the hyperkähler manifolds.

In this paper, we will study the case of higher $n$ and $k$. We construct examples and prove some structure results. If $O_X$ is a $\mathbb{P}^n[k]$-object with $k > 2$, the manifold $X$ is automatically projective; see Lemma 4.3.10. Hence, we will call $X$ a variety with $\mathbb{P}^n[k]$-unit. The main results of this paper can be summarised as

Theorem 4.1.3. Let $n + 1 = p^r$ be a prime power. Then the following are equivalent:

1. There exists a variety with $\mathbb{P}^n[4]$-unit,

2. There exists a variety with $\mathbb{P}^n[k]$-unit for every even $k$,

3. There exists a strict Enriques variety of index $n + 1$.

For $n + 1$ arbitrary, the implications (iii)$\Rightarrow$(ii)$\Rightarrow$(i) are still true.

We do not know whether or not (i)$\Rightarrow$(iii) is true in general if $n + 1$ is not a prime power, but we will prove a slightly weaker statement that holds for arbitrary $n + 1$; see Section 4.5.2. In particular, the universal cover of a variety with $\mathbb{P}^n[4]$-unit, with $n + 1$ arbitrary, splits into a product of two hyperkähler varieties; see Proposition 4.5.3.

Our notion of strict Enriques varieties is inspired by similar notions of higher dimensional analogues of Enriques surfaces due to Boissière, Nieper-Wißkirchen, and Sarti [BNS11] and Oguiso and Schröer [OS11]. There are known examples of strict Enriques varieties of index 3 and 4. Hence, we get

Corollary 4.1.4. For $n = 2$ and $n = 3$ there are examples of varieties with $\mathbb{P}^n[k]$-units for every even $k \in \mathbb{N}$.
The motivation for this work comes from questions concerning derived categories and the notions are influenced by this. However, in this paper, with the exception Sections 4.6.5 and 4.6.6, all results and proofs are also formulated without using the language of derived categories.

The paper is organised as follows. In Section 4.2.1, we fix some notations and conventions. Sections 4.2.2 and 4.2.3 are a very brief introduction into derived categories and some types of objects that occur in these categories. In particular, we introduce the notion of $\mathbb{P}^n[k]$-objects.

In Section 4.3.1, we say a few words about compact hyperkähler manifolds. In Section 4.3.2, we discuss automorphisms of Beauville–Bogomolov products and their action on cohomology. This is used in the following Section 4.3.3 in order to give a proof of Observation 4.1.2. This proof is probably a bit easier than the one in [HN11, App. A]. More importantly, it allows us to introduce some of the notations and ideas which are used in the later sections. In Section 4.3.4, we discuss a class of varieties which we call strict Enriques varieties. There are two different notions of Enriques varieties in the literature (see [BNS11] and [OS11]) and our notion is the intersection of these two; see Proposition 4.3.144. In Section 4.3.5, we quickly mention a generalisation; namely strict Enriques stacks.

We give the definition of a variety with a $\mathbb{P}^n[k]$-unit together with some basic remarks in Section 4.4.1. Section 4.4.2 provides two examples of varieties which look like promising candidates, but ultimately fail to have $\mathbb{P}^n[k]$-units. In Section 4.4.3, we construct series of varieties with $\mathbb{P}^n[k]$-units out of strict Enriques varieties of index $n + 1$. In particular, we prove the implication $(iii) \implies (ii)$ of Theorem 4.1.3.

In Section 4.5.1, we make some basic observations concerning the fundamental group and the universal cover of varieties with $\mathbb{P}^n[k]$-units. In Section 4.5.2, we specialise to the case $k = 4$. We proof that the universal cover of a variety with $\mathbb{P}^n[4]$-unit is the product of two hyperkähler manifolds of dimension $2n$. Then we proceed to prove the implication $(i) \implies (iii)$ of Theorem 4.1.3 for $n + 1$ a prime power.

Section 4.6 is a collection of some further observations and ideas. In Sections 4.6.1, 4.6.2, and 4.6.3, some further constructions leading to varieties with $\mathbb{P}^n[k]$-units are discussed. We talk briefly about stacks with $\mathbb{P}^n[k]$-units in Section 4.6.4. In Section 4.6.5, we prove that the class of strict Enriques varieties is stable under derived equivalences, and in Section 4.6.6 we study some derived autoequivalences of varieties with $\mathbb{P}^n[k]$-units. In the final Section 4.6.7, we contemplate a bit about varieties with $\mathbb{P}^n[k]$-units as moduli spaces and constructions of hyperkähler varieties.

Acknowledgements. The early stages of this work were done while the author was financially supported by the research grant KR 4541/1-1 of the DFG (German Research Foundation). He thanks Daniel Huybrechts, Marc Nieper-Wißkirchen, Sonke Rollenske, and Pawel Sosna for helpful discussions and comments. He also thanks the referee for helpful comments and suggestions.

4.2 Notations and preliminaries

4.2.1 Notations and conventions

1. Throughout, $X$ will be a connected compact Kähler manifold (often a smooth projective variety).

2. We denote the universal cover by $\tilde{X} \to X$. 

108
3. If \( \omega_X \) is of finite order \( m \), we denote the canonical cover by \( \pi: \tilde{X} \to X \). It is defined by the properties that \( \omega_X \) is trivial and \( \pi \) is an étale Galois cover of degree \( m \). We have \( \pi_* \mathcal{O}_{\tilde{X}} \cong \mathcal{O}_X \oplus \omega_X^{-1} \oplus \omega_X^{-2} \oplus \cdots \oplus \omega_X^{-(m-1)} \) and the covering map \( \tilde{X} \to X \) is the quotient by a cyclic group \( G = \langle g \rangle \) with \( g \in \text{Aut}(\tilde{X}) \) of order \( m \).

4. We will usually write graded vector spaces in the form \( V^* = \oplus_{i \in \mathbb{Z}} V^i[-i] \). The Euler characteristic is given by the alternating sum \( \chi(V^*) = \sum_{i \in \mathbb{Z}} (-1)^i \dim V^i \).

5. Given a sheaf or a complex of sheaves \( E \) and an integer \( i \in \mathbb{Z} \), we write \( H^i(X, E) \) for the \( i \)-th derived functor of global sections. In contrast, \( \mathcal{H}^i(E) \) denotes the cohomology of the complex in the sense kernel modulo image of the differentials.

6. We write for short \( Y \in \text{HK}_{2d} \) to express the fact that \( Y \) is a compact hyperkähler manifold of dimension \( 2d \). In this case, we denote by \( y \) a generator of \( \text{H}^2(\mathcal{O}_Y) \), i.e. \( y \) is the complex conjugate of a symplectic form on \( Y \). If we just write \( Y \in \text{HK} \), this means that \( Y \) is a hyperkähler manifold of unspecified dimension. Sometimes, we write \( Y \in \text{K3} \) instead of \( Y \in \text{HK}_2 \).

7. We write for short \( Z \in \text{CY}_e \) to express the fact that \( Z \) is a compact simply connected strict Calabi–Yau variety of dimension \( e \geq 3 \). In this case, we denote by \( z \) a generator of \( \mathcal{H}^e(\mathcal{O}_Z) \), i.e. \( z \) is the complex conjugate of a volume form on \( Z \). If we just write \( Z \in \text{CY} \), this means that \( Z \) is a simply connected strict Calabi–Yau variety of unspecified dimension.

8. We denote the connected zero-dimensional manifold by \( \text{pt} \).

9. For \( n \in \mathbb{N} \), we denote the symmetric group of permutations of the set \( \{1, \ldots, n\} \) by \( \mathfrak{S}_n \). Given a space \( X \) and a permutation \( \sigma \in \mathfrak{S}_n \), we denote the automorphism of the cartesian product \( X^n \) which is given by the according permutation of components again by \( \sigma \in \text{Aut}(X^n) \).

10. The symbol \( \sum_{i_1 \neq i_2 \neq \cdots \neq i_\ell} \) means summation over sets \( \{i_1, \ldots, i_\ell\} \) of cardinality \( \ell \) (contained in some fixed index set which is, hopefully, clear from the context)

### 4.2.2 Derived categories of coherent sheaves

As mentioned in the introduction, knowledge of derived categories is not necessary for the understanding of this paper. However, often things can be stated in the language of derived categories in the most convenient way, and questions concerning derived categories motivated this work. Hence, we will give, in a very brief form, some basic definitions and facts.

The derived category \( \text{D}(X) := \text{D}^b(\text{Coh}(X)) \) is defined as the localisation of the homotopy category of bounded complexes of coherent sheaves by the class of quasi-isomorphisms. Hence, the objects of \( \text{D}(X) \) are bounded complexes of coherent sheaves. The morphisms are morphisms of complexes together with formal inverses of quasi-isomorphisms. In particular, every quasi-isomorphism between complexes becomes an isomorphism in \( \text{D}(X) \). The derived category \( \text{D}(X) \) is a triangulated category. In particular, there is the shift autoequivalence \([1]: \text{D}(X) \to \text{D}(X) \)\).

Given two objects \( E, F \in \text{D}(X) \), there is a graded Hom-space \( \text{Hom}^*(E, F) = \bigoplus \text{Hom}^i_{\text{D}(X)}(E, F[i])[-i] \). For \( E = F \), this is a graded algebra by the Yoneda product (composition of morphisms). There is a fully faithful embedding \( \text{Coh}(X) \hookrightarrow \text{D}(X) \),
\[ A \mapsto A[0] \] which is given by considering sheaves as complexes concentrated in degree zero. Most of the time, we will denote \( A[0] \) simply by \( A \) again. For \( A, B \in \text{Coh}(X) \), we have \( \text{Hom}^*(A, B) \cong \text{Ext}^*(A, B) \). Besides the shift functor, the data of a triangulated category consists of a class of distinguished triangles \( E \to F \to G \to E[1] \) consisting of objects and morphisms in \( D(X) \) satisfying certain axioms. In particular, every morphism \( f: E \to F \) in \( D(X) \) can be completed to a distinguished triangle

\[ E \xrightarrow{f} F \to G \to E[1]. \]

The object \( G \) is determined by \( f \) up to isomorphism and denoted by \( G = \text{cone}(f) \). There is a long exact cohomology sequence

\[ \cdots \to H^{i-1}(\text{cone}(f)) \to H^i(E) \to H^i(F) \to H^i(\text{cone}(f)) \to H^{i+1}(E) \to \cdots. \]

### 4.2.3 Special objects of the derived category

In the following, we will recall the notions of exceptional, spherical and \( \mathbb{P} \)-objects in the derived category \( D(X) \) of coherent sheaves on a compact Kähler manifold \( X \). Exceptional objects can be used in order to decompose derived categories while spherical and \( \mathbb{P} \)-objects induce autoequivalences; see also Section 4.6.6. Our main focus in this paper, however, will be to characterise varieties where \( O_X \in D(X) \) is an object of one of these types.

**Definition 4.2.1.** An object \( E \in D(X) \) is called **exceptional** if \( \text{Hom}^*(E, E) \cong \mathbb{C}[0] \).

Let \( X \) be a Fano variety, i.e. the anticanonical bundle \( \omega_X^{-1} \) is ample. Then, by Kodaira vanishing, every line bundle on \( X \) is exceptional when considered as an object of the derived category \( D(X) \); see also Remark 4.2.8. Similarly, every line bundle on an Enriques surface is exceptional. Another typical example of an exceptional object is the structure sheaf \( O_C \in D(S) \) of a \((-1)\)-curve \( \mathbb{P}^1 \cong C \subset S \) on a surface.

**Definition 4.2.2 ([ST01]).** An object \( E \in D(X) \) is called **spherical** if

1. \( E \otimes \omega_X \cong E \),
2. \( \text{Hom}^*(E, E) \cong \mathbb{C}[0] \oplus \mathbb{C}[\dim X] \cong H^*(S^{\dim X}, \mathbb{C}) \).

Every line bundle on a strict Calabi–Yau variety is spherical. Another typical example of a spherical object is the structure sheaf \( O_C \in D(S) \) of a \((-2)\)-curve \( \mathbb{P}^1 \cong C \subset S \) on a surface.

**Definition 4.2.3 ([HT06]).** Let \( n \in \mathbb{N} \). An object \( E \in D(X) \) is called **\( \mathbb{P}^n \)-object** if

1. \( E \otimes \omega_X \cong E \),
2. There is an isomorphism of \( \mathbb{C} \)-algebras \( \text{Hom}^*(E, E) \cong \mathbb{C}[x]/x^{n+1} \) with \( \deg x = 2 \).

Condition (ii) can be rephrased as \( \text{Hom}^*(E, E) \cong H^*(\mathbb{P}^n, \mathbb{C}) \). As we will see in the next subsection, every line bundle on a compact hyperkähler manifold is a \( \mathbb{P} \)-object. Another typical example is the structure sheaf of the centre of a Mukai flop.

**Definition 4.2.4.** Let \( n, k \in \mathbb{N} \). An object \( E \in D(X) \) is called **\( \mathbb{P}^n[k] \)-object** if

1. \( E \otimes \omega_X \cong E \),

110
2. There is an isomorphism of \( \mathbb{C} \)-algebras \( \text{Hom}^*(E, E) \cong \mathbb{C}[x]/x^{n+1} \) with \( \deg x = k \).

**Remark 4.2.5.** If there is a \( \mathbb{P}^n[k] \)-object \( E \in D(X) \), we have \( \dim X = n \cdot k \) by Serre duality.

**Remark 4.2.6.** For \( n = 1 \), the \( \mathbb{P}^1[k] \)-objects coincide with the spherical objects. For \( k = 2 \), the \( \mathbb{P}^n[2] \)-objects are exactly the \( \mathbb{P}^n \)-objects in the sense of Huybrechts and Thomas.

The names spherical and \( \mathbb{P} \)-objects come from the fact that their graded endomorphism algebra coincides with the cohomology of spheres and projective spaces, respectively. Hence, it would be natural to name \( \mathbb{P}^n[k] \)-object by series of manifolds whose cohomology is of the form \( \mathbb{C}[x]/x^{n+1} \) with \( \deg x = k \). For \( k = 4 \), there are the quaternionic projective spaces. For \( k > 4 \), however, there are probably no such series. At least, there are no manifolds \( M \) satisfying the possibly stronger condition that \( H^*(M, \mathbb{Z}) \cong \mathbb{Z}[x]/x^{n+1} \) for \( \deg x > 4 \) and \( n > 2 \); see [Hat02, Cor. 4L.10]. Hence, we will stick to the notion of \( \mathbb{P}^n[k] \)-objects which is justified by the following

**Remark 4.2.7.** A \( \mathbb{P}^n[k] \)-object is essentially the same as a \( \mathbb{P} \)-functor (see [Add16]) \( D(\text{pt}) \to D(X) \) with \( \mathbb{P} \)-cotwist \( [-k] \). In particular, as we will further discuss in Section 4.6.6, it induces an autoequivalence of \( D(X) \).

**Remark 4.2.8.** Given a compact Kähler manifold \( X \), the following are equivalent:

1. \( \mathcal{O}_X \) is a \( \mathbb{P}^n[k] \)-object.
2. Every line bundle on \( X \) is a \( \mathbb{P}^n[k] \)-object.
3. Some line bundle on \( X \) is a \( \mathbb{P}^n[k] \)-object.

The same holds if we replace the property to be a \( \mathbb{P}^n[k] \)-object by the property to be an exceptional object. Indeed, for every line bundle \( L \) on \( X \), we have isomorphisms of \( \mathbb{C} \)-algebras

\[
\text{Hom}^*(L, L) \cong \text{Hom}^*(\mathcal{O}_X, \mathcal{O}_X) \cong H^*(\mathcal{O}_X)
\]

where the latter is an algebra by the cup product. Furthermore, \( L \otimes \omega_X \cong L \) holds if and only if \( \omega_X \) is trivial.

### 4.3 Hyperkähler and Enriques varieties

In this section, we first review some results on hyperkähler manifolds and their automorphisms. In particular, we give a proof of Observation 4.1.2, i.e. the fact that hyperkähler manifolds can be characterised by the property that the trivial line bundle is a \( \mathbb{P} \)-object. Then we introduce and study strict Enriques varieties. They are a generalisation of Enriques surfaces to higher dimensions and can be realised as quotients of hyperkähler varieties.

#### 4.3.1 Hyperkähler manifolds

Let \( X \) be a compact Kähler manifold of dimension \( 2n \). We say that \( X \) is hyperkähler if and only if its Riemannian holonomy group is the symplectic group \( \text{Sp}(n) \). A compact Kähler manifold \( X \) is hyperkähler if and only if it is *irreducible holomorphic symplectic* which means that it is simply connected and \( H^2(X, \Omega_X^2) \) is spanned by an everywhere non-degenerate 2-form, called *symplectic form*; see e.g. [Bea83] or [Huy03].
The structure sheaf of a hyperkähler manifold is a $\mathbb{P}^n$-object; see [HT06, Ex. 1.3(ii)]. This means that the canonical bundle $\omega_X = \Omega_X^{2n}$ is trivial and $H^*(\mathcal{O}_X) = \mathbb{C}[x]/x^{n+1}$; compare Item 6 of Section 4.2.1. This follows essentially from the holonomy principle together with Bodner’s principle. We will see in Section 4.3.3 that also the converse holds, which amounts to Observation 4.1.2.

### 4.3.2 Automorphisms and their action on cohomology

In the later sections, we will often deal with automorphisms of Beauville–Bogomolov covers. There is the following result of Beauville [Bea83, Sect. 3].

**Lemma 4.3.1.** Let $X' = \prod_i Y_i^{\lambda_i} \times \prod_j Z_j^{\nu_j}$ be a finite product with $Y_i \in \text{HK}_{2d_i}$ and $Z_j \in \text{CY}_{e_j}$ such that the $Y_i$ and $Z_j$ are pairwise non-isomorphic. Then, every automorphism of $X'$ preserves the decomposition up to permutation of factors. More concretely, every automorphism $f \in \text{Aut}(X')$ is of the form $f = \prod_i f_{Y_i}^{\lambda_i} \times \prod_j f_{Z_j}^{\nu_j}$ with $f_{Y_i}^{\lambda_i} \in \text{Aut}(Y_i^{\lambda_i})$ and $f_{Z_j}^{\nu_j} \in \text{Aut}(Z_j^{\nu_j})$. Furthermore, $f_{Y_i}^{\lambda_i} = (f_{Y_1}^{\lambda_1} \times \cdots \times f_{Y_{\mu_i}}^{\lambda_{\mu_i}}) \circ \sigma_{Y_i}^{f}$ with $f_{Y_i}^{\lambda_i} \in \text{Aut}(Y_i)$ and $\sigma_{Y_i}^{f} \in \mathcal{S}_{\lambda_i}$. Similarly, $f_{Z_j}^{\nu_j} = (f_{Z_1}^{\nu_1} \times \cdots \times f_{Z_{\nu_j}}^{\nu_{\nu_j}}) \circ \sigma_{Z_j}^{f}$ with $f_{Z_j}^{\nu_j} \in \text{Aut}(Z_j)$ and $\sigma_{Z_j}^{f} \in \mathcal{S}_{\nu_j}$. Let $Y \in \text{HK}$. The action of automorphisms on $H^2(\mathcal{O}_Y) \cong \mathbb{C}$ defines a group character which we denote by $\varrho_Y: \text{Aut}(Y) \to \mathbb{C}^* \; , \; f \mapsto \varrho_Y(f)$. In particular, an automorphism $f \in \text{Aut}(Y)$ of finite order $\text{ord} f = m$ acts on $H^2(\mathcal{O}_Y)$ by multiplication by an $m$-th root of unity $\varrho_Y(f) \in \mu_m$. Similarly, for $Z \in \text{CY}_k$ we have a character $\varrho_Z: \text{Aut}(Z) \to \mathbb{C}^*$ given by the action of automorphisms on $H^k(\mathcal{O}_Z)$.

**Corollary 4.3.2.** Let $f \in \text{Aut}(X')$ be of finite order $d$. Then the induced action of $f$ on the cohomology of the structure sheaf is given by permutations of the $y_{ia}$ with fixed $i$ and the $z_{j\beta}$ with fixed $j$ together with multiplications by $d$-th roots of unity. This means

$$f: \; y_{ia} \mapsto \varrho_{Y_{ia}} \cdot y_{ia}, \; z_{j\beta} \mapsto \varrho_{Z_{j\beta}} \cdot z_{j\beta}, \; \sigma_{Y_{ia}} \mapsto \varrho_{Y_{ia}}^{d_i}, \; \sigma_{Z_{j\beta}} \mapsto \varrho_{Z_{j\beta}}^{d_j}$$

with $\varrho_{Y_{ia}}, \varrho_{Z_{j\beta}} \in \mu_d$.

The main takeaway for the computations in the latter sections is that the cohomology classes can only be permuted if the corresponding factors of the product coincide.

**Definition 4.3.3.** Let $Y \in \text{HK}$ and $f \in \text{Aut} Y$ of finite order. We call the order of $\varrho_Y(f) \in \mathbb{C}$ the symplectic order of $f$. The reason for the name is that $f$ acts by a root of unity of the same order, namely $\varrho_Y(f)$, on $H^0(\Omega_Y^2)$, i.e. on the symplectic forms. In general, the symplectic order divides the order of $f$ in $\text{Aut}(X)$. We say that $f$ is purely non-symplectic if its symplectic order is equal to $\text{ord} f$. [112]
Lemma 4.3.4. Let $Y \in \text{HK}_{2n}$ and let $f \in \text{Aut}(Y)$ be an automorphism of finite order $m$ such that the generated group $\langle f \rangle$ acts freely on $Y$. Then $f$ is purely non-symplectic and $m \mid n + 1$. Similarly, every fixed point free automorphism of finite order of a strict Calabi–Yau variety is a non-symplectic involution.

Proof. This follows from the holomorphic Lefschetz fixed point theorem; compare [BNS11, Sect. 2.2].

Corollary 4.3.5. Let $Y \in \text{HK}_{2n}$ and let $X = Y/\langle f \rangle$ be the quotient by a cyclic group of automorphisms acting freely. Then $\omega_X$ is non-trivial and of finite order.

Proof. The order of $\omega_X$ is exactly the order of the action of $f$ on $H^n(O_Y)$, i.e. the order of $\varphi^n_{Y,f} \in \mathbb{C}^*$. By the previous lemma, this order is finite and greater than one.

Here is a simple criterion for automorphisms of products to be fixed point free.

Lemma 4.3.6.

1. Let $X_1, \ldots, X_k$ be manifolds and $f_i \in \text{Aut}(X_i)$. Then $f_1 \times \cdots \times f_k \in \text{Aut}(X_1 \times \cdots \times X_k)$

is fixed point free if and only if at least one of the $f_i$ is fixed point free.

2. Let $X$ be a manifold and $g_1, \ldots, g_k \in \text{Aut}(X)$. Consider the automorphism

$$\varphi = (g_1 \times \cdots \times g_k) \circ (1 \ 2 \ \ldots \ k) \in \text{Aut}(X^k)$$

given by $(p_1, p_2, \ldots, p_k) \mapsto (g_1(p_k), g_2(p_1), \ldots, g_k(p_{k-1}))$. Then $\varphi$ is fixed point free if and only if the composition $g_i \circ g_{i-1} \circ \cdots \circ g_1$ (or, equivalently, $g_i \circ g_{i-1} \circ \cdots \circ g_{i+1}$ for some $i = 1, \ldots, k$) is fixed point free.

We also will frequently use the following well-known fact.

Lemma 4.3.7. Let $X'$ be a smooth projective variety and let $G \subset \text{Aut}(X')$ be a finite subgroup which acts freely. Then, the quotient variety $X := X'/G$ is again smooth projective and

$$\chi(O_{X'}) = \chi(O_X) \cdot \text{ord } G.$$

Furthermore, $H^*(O_X) = H^*(O_X')^G$.

4.3.3 Proof of Observation 4.1.2

We already remarked in Section 4.3.1 that the structure sheaf of a hyperkähler manifold is a $\mathbb{P}$-object. Hence, for the verification of Observation 4.1.2 we only need to prove the following

Proposition 4.3.8. Let $X$ be a compact Kähler manifold such that $O_X \in \mathcal{D}(X)$ is a $\mathbb{P}^n[2]$-object. Then $X$ is hyperkähler of dimension $2n$. 

Proof. As already mentioned in the introduction, this follows immediately from [HN11, Prop. A.1]. We will give a slightly different proof.

Recall that the assumption that $O_X$ is a $\mathbb{P}$-object means
1. $\omega_X$ is trivial,

2. $H^*(\mathcal{O}_X) \cong \mathbb{C}[x]/x^{n+1}$ with $\deg x = 2$.

For $n = 1$, it follows easily by the Kodaira classification of surfaces, that $X \in K3 = HK_2$. Hence, we may assume that $n \geq 2$.

Assumption (i) says that, in particular, $c_1(X) = 0$. Hence, we have a finite étale covering $X' \rightarrow X$ and a Beauville–Bogomolov decomposition

$$X' = T \times \prod_i Y_i \times \prod_j Z_j.$$ \hfill (4.2)

The plan is to show that $X'$ is hyperkähler and the covering map is an isomorphism.

**Convention 4.3.9.** Whenever we have a Beauville–Bogomolov decomposition of the form (4.2), $T$ is a complex torus, $Y_i \in HK_{2d_i}$ is a hyperkähler of dimension $2d_i$ and $Z_j \in CY_{e_j}$ is a strict simply connected Calabi–Yau variety of dimension $e_j \geq 3$. Furthermore, $H^2(\mathcal{O}_{Y_i}) = \langle y_i \rangle$ and $H^{e_j}(\mathcal{O}_{Z_j}) = \langle z_j \rangle$.

By Assumption (ii), we have $\chi(\mathcal{O}_X) = n + 1$. On the other hand, since $X' \rightarrow X$ is étale, say of degree $m$, we have

$$m(n + 1) = m \cdot \chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X'}) = \chi(T) \cdot \prod_i \chi(\mathcal{O}_{Y_i}) \cdot \prod_j \chi(\mathcal{O}_{Z_j}).$$ \hfill (4.3)

This implies that $T = pt$ and all $e_j$ are even. Otherwise, the right-hand side of (4.3) would be zero. Since the torus part is trivial, $X'$ is simply connected. Hence, $X' = \tilde{X}$ is the universal cover of $X = \tilde{X}/G$ where $\pi_1(X) \cong G \subset \text{Aut}(X)$. It follows by Lemma 4.3.7 that

$$\mathbb{C}[x]/x^{n+1} \cong H^*(\mathcal{O}_X) \cong H^*(\mathcal{O}_{\tilde{X}})^G \subset H^*(\mathcal{O}_{\tilde{X}}).$$

In particular, there must be an $x \in H^2(\mathcal{O}_{\tilde{X}})^G \subset H^2(\mathcal{O}_{\tilde{X}})$ such that $x^n \neq 0$ in $H^{2n}(\mathcal{O}_{\tilde{X}})$. Since

$$2n = \dim X = \dim \tilde{X} = \sum_i 2d_i + \sum_j e_j,$$

we have $H^{2n}(\mathcal{O}_{\tilde{X}}) = \langle s \rangle$ with $s = \prod_i y_i^{d_i} \cdot \prod_j z_j$. As $\deg x = 2$ and $\deg z_j = e_j \geq 3$, it follows that $x$ is a linear combination of some $y_i$. Hence, $x^n$ can be a non-zero multiple of $s$ only if no $z_j$ occurs in the expression of $s$ as above. In other words, $X'$ does not have Calabi–Yau factors. This means that $X = \prod_i Y_i$ and $x = \sum_i y_i$ (up to coefficients which we can absorb by the choice of the generators $y_i$ of $H^3(\mathcal{O}_{Y_i})$). Every element of $G$ acts by some permutation on the $y_i$; see Corollary 4.3.2. By assumption, $H^2(\mathcal{O}_X)$ is of dimension one. Hence, $H^2(\mathcal{O}_{\tilde{X}})^G = \langle x \rangle$. It follows that the action of $G$ on the $y_i$ is transitive. Otherwise, there would be $G$-invariant summands of $x = \sum y_i$ which would be linearly independent. Hence, again by Corollary 4.3.2, we have $\tilde{X} \cong Y^\ell$ for some $Y \in HK_{2d}$. For dimension reasons, $d \cdot \ell = n$.

We assume for a contradiction that $\ell > 1$. We have the $G$-invariant class

$$x^2 = \sum_\alpha y_\alpha^2 + 2 \sum_{\alpha \neq \beta} y_\alpha y_\beta \in H^4(\mathcal{O}_{X'})^G = H^4(\mathcal{O}_X).$$ \hfill (4.4)
It follows by Corollary 4.3.2 that the two summands in (4.4) are again \( G \)-invariant. But, by assumption, \( h^4(\mathcal{O}_X) = 1 \). Thus, one of the two summands must be zero. By (4.1), we see that the only possibility for this to happen is \( d = 1 \), i.e. \( Y \in \text{K3} \). Thus, \( \ell = n \). Note that \( \text{ord} G = \text{deg}(X' \to X) = m \). By (4.3) or Lemma 4.3.7, we have \( m \mid \chi(X') = \chi(Y)^n = 2^n \). As \( G \) acts transitively on \( \{y_1, \ldots, y_n\} \) we get \( n \mid m \mid 2^n \). Again by (4.3), also \( n + 1 \mid 2^n \). For \( n \geq 2 \), this is a contradiction.

Hence, we are in the case \( \ell = 1 \) which means that \( \hat{X} = Y \in \text{HK}_{2n} \). In particular, \( \chi(\mathcal{O}_{\hat{X}}) = n + 1 = \chi(X) \). By (4.3), we get \( m = 1 \) which means that we have an isomorphism \( Y \cong X \).

### 4.3.4 Enriques varieties

In this section we will consider a certain class of compact Kähler manifolds with the property that \( \mathcal{O}_X \in \mathbb{D}(X) \) is exceptional; see Definition 4.2.1. These manifolds are automatically algebraic by the following result; see e.g. [Voi07, Exc. 7.1].

**Lemma 4.3.10.** Let \( X \) be a compact Kähler manifold with \( \mathcal{H}^2(\mathcal{O}_X) = 0 \). Then \( X \) is projective.

From now on, let \( E \) be a smooth projective variety.

**Definition 4.3.11.** We call \( E \) a strict Enriques variety if the following three conditions hold:

(S1) The trivial line bundle \( \mathcal{O}_E \) is exceptional.

(S2) The canonical line bundle \( \omega_E \) is non-trivial and of finite order \( m := \text{ord}(\omega_E) \) in \( \text{Pic} E \) (this order is called the index of \( E \)).

(S3) The canonical cover \( \tilde{E} \) of \( E \) is hyperkähler.

This definition is inspired by similar, but different, notions of higher-dimensional Enriques varieties which are as follows.

**Definition 4.3.12 ([BNS11]).** We call \( E \) a BNWS (Boissière-Nieper-Wißkirchen-Sarti) Enriques variety if the following three conditions hold:

(BNWS1) \( \chi(\mathcal{O}_E) = 1 \).

(BNWS2) The canonical line bundle \( \omega_E \) is non-trivial and of finite order \( m := \text{ord}(\omega_E) \) in \( \text{Pic} E \) (this order is called the index of \( E \)).

(BNWS3) The fundamental group of \( E \) is cyclic of the same order, i.e. \( \pi_1(E) \cong \mu_m \).

**Definition 4.3.13 ([OS11]).** We call \( E \) an OS (Oguiso-Schröer) Enriques variety if \( E \) is not simply connected and its universal cover \( \tilde{E} \) is a compact hyperkähler manifold.

**Proposition 4.3.14.**

1. Let \( E \) be a strict Enriques variety of index \( n + 1 \). Then \( \dim E = 2n \).

2. Conversely, every smooth projective variety \( E \) satisfying (S2) with \( m = n + 1 \), (S3), and \( \dim E = 2n \) is already a strict Enriques variety.
3. Strict Enriques varieties of index $n + 1$ are exactly the quotient varieties of the form $E = Y/\langle g \rangle$, where $Y \in \text{HK}_{2n}$ and $g \in \text{Aut}(Y)$ is purely non-symplectic of order $n + 1$ such that $\langle g \rangle$ acts freely on $Y$.

4. $X$ is a strict Enriques variety if and only if it is BNWS Enriques and OS Enriques.

Proof. Let $E$ be a strict Enriques variety of index $n + 1$ with canonical cover $\tilde{E} \in \text{HK}_{2d}$. To verify 1 we have to show that $d = n$. By definition of the canonical cover (see Section 4.2.1 3), the covering map $\tilde{E} \to E$ is the quotient by a cyclic group $G$ of order $n + 1$. As $\tilde{E} \in \text{HK}_{2d}$, we have $\chi(\mathcal{O}_E) = d + 1$. Also, $\chi(\mathcal{O}_E) = 1$ by (S1). We get $d = n$ by Lemma 4.3.7.

Consider now a smooth projective variety $E$ with $\text{ord} \omega_E = n + 1$ and $\dim E = 2n$ such that its canonical cover $\tilde{E}$ is hyperkähler, necessarily of dimension $\dim \tilde{E} = \dim E = 2n$. Then, again by Lemma 4.3.7, we have $\chi(\mathcal{O}_E) = 1$. Furthermore,

$$\mathbb{C}[0] \subset H^*(\mathcal{O}_E) \cong H^*(\mathcal{O}_\tilde{E})^{\mu_{n+1}} \subset H^*(\mathcal{O}_\tilde{E}) \cong \mathbb{C}[y]/y^{n+1} \quad (4.5)$$

with $\deg y = 2$. Since $\chi(\mathcal{O}_E) = 1$ and $H^*(\mathcal{O}_\tilde{E})$ is concentrated in even degrees, the first inclusion must be an equality which means that $\mathcal{O}_E$ is exceptional.

Let us proof part 3. Given a strict Enriques variety $E$ of index $n + 1$ the canonical cover $Y := \tilde{E}$ has the desired properties.

Conversely, let $Y \in \text{HK}_{2n}$ together with a purely non-symplectic $g \in \text{Aut}(Y)$ of order $n + 1$ such that $\langle g \rangle$ acts freely on $Y$, and set $E := Y/\langle g \rangle$. The action of $g$ on the cohomology $H^*(\mathcal{O}_Y) = \mathbb{C}[y]/y^{n+1}$ is given by $g \cdot y^i = g_Y^i y^i$. Since, by assumption, $g_Y$ is a primitive $(n + 1)$-th root of unity, we get $H^*(\mathcal{O}_E) \cong H^*(\mathcal{O}_Y)^G \cong \mathbb{C}[0]$, hence (S1). The action of $g$ on the $n$-th power of a symplectic form, hence on the canonical bundle $\omega_Y$, is also given by multiplication by $g_Y$. It follows that the canonical bundle $\omega_E$ of the quotient is of order $n + 1$ and $Y \to E$ is the canonical cover.

For the proof of 4, first note that (S1) implies (BNWS1). Furthermore, given a strict Enriques variety $E$, the canonical cover $Y := \tilde{E}$ of $E$ is also the universal cover, since $Y$ is connected. From this, we get (BNWS2) and (BNWS3). Furthermore, $E$ is OS Enriques, since $Y$ is hyperkähler.

Conversely, if $E$ is BNWS and OS Enriques, its canonical and universal cover coincide and is given by a hyperkähler manifold $Y$ with the properties as in 3.

Note that the variety $Y \in \text{HK}_{2n}$ from part 3 of the proposition is the universal as well as the canonical cover of $E$. We call $Y$ the hyperkähler cover of $E$.

Another way to characterise strict Enriques varieties is as OS Enriques varieties whose fundamental group have the maximal possible order; see [OS11, Prop. 2.4].

Strict Enriques varieties of index 2 are exactly the Enriques surfaces. To get examples of higher index, by part 4 of the previous proposition, we just have to look for examples which occur in [BNS11] as well as in [OS11].

**Theorem 4.3.15** ([BNS11],[OS11]). There are strict Enriques varieties of index 2, 3, and 4.

Note that the statement does not exclude the existence of strict Enriques varieties of index greater than 4, but, for the time being, there are no known examples.

In the known examples of index $n + 1 = 3$ or $n + 1 = 4$, the hyperkähler cover $Y$ is given by a generalised Kummer variety $K_nA \subset A^{[n+1]}$. More concretely, in these examples $A$ is an abelian surface isogenous to a product of elliptic curves with complex multiplication, and there
is a non-symplectic automorphism \( f \in \operatorname{Aut}(A) \) of order \( n + 1 \) which induces a non-symplectic fixed point free automorphism \( K_n(f) \in \operatorname{Aut}(K_nA) \) of the same order.

Note that there are examples of varieties which are BNWS Enriques but not OS Enriques [BNS11, Sect. 4.3] and of the converse [OS11, Sect. 4].

We will use the following lemma in the proof of Theorem 4.4.5.

**Lemma 4.3.16.** Let \( E \) be a strict Enriques variety of index \( n + 1 \) with hyperkähler cover \( Y \). Then there is an isomorphism of algebras \( \oplus_{s=0}^{n} H^s(\omega_{E}^{-s}) \cong H^*(\mathcal{O}_Y) = \mathbb{C}[y]/y^{n+1} \). Under this isomorphism, \( H^*(\omega_{E}^{-s}) \cong \mathbb{C} \cdot y^s \cong \mathbb{C}[-2s] \).

**Proof.** Let \( \pi: Y \to E \) be the morphism which realises \( Y \) as the universal and canonical cover of \( E \). By the construction of the canonical cover (see Section 4.2.13), we have an isomorphism of \( \mathcal{O}_E \)-algebras \( \pi_* \mathcal{O}_Y \cong \mathcal{O}_E \oplus \omega_E^{-1} \oplus \cdots \oplus \omega_E^{-n} \). Hence, we get an isomorphism of graded \( \mathbb{C} \)-algebras

\[
\mathbb{C}[y]/y^{n+1} \cong H^*(\mathcal{O}_Y) \cong H^*(\mathcal{O}_E) \oplus H^*(\omega_E^{-1}) \oplus \cdots \oplus H^*(\omega_E^{-n}) \tag{4.6}
\]

with \( \deg y = 2 \). Hence, for the proof of the assertion, it is only left to show that the generator \( y \) lives in the direct summand \( H^2(\omega_E^{-1}) \) under the decomposition (4.6). We have \( y \in H^2(\omega_E) \) for some \( s \in \mathbb{Z}/(n+1)\mathbb{Z} \). By Serre duality, we have \( H^*(\omega_E) = \mathbb{C}[-2n] \), hence \( y^n \in H^{2n}(\omega_E) \). It follows that

\[-s \equiv n \cdot s \equiv 1 \mod n + 1. \]

\( \square \)

### 4.3.5 Enriques stacks

The main difficulty in finding pairs \( Y \in \text{HK} \) and \( f \in \operatorname{Aut}(Y) \) which, by Proposition 4.3.143, induce strict Enriques varieties, is the condition that \( (f) \) acts freely.

Let us drop this assumption and consider a \( Y \in \text{HK}_{2n} \) together with a purely non-symplectic automorphism \( f \in \operatorname{Aut}(Y) \) which may have fixed points. Then we call the corresponding quotient stack \( \mathcal{E} = [Y/(f)] \) a strict Enriques stack. In analogy to the proof of Proposition 4.3.14, one can show that there is also the following equivalent

**Definition 4.3.17.** A strict Enriques stack is a smooth projective stack \( \mathcal{E} \) such that

(S1') The trivial line bundle \( \mathcal{O}_E \) is exceptional.

(S2') The canonical bundle \( \omega_E \) is non-trivial and of finite order \( m := \operatorname{ord}(\omega_E) \) in \( \text{Pic} \mathcal{E} \) (this order is called the index of \( \mathcal{E} \)).

(S3') The canonical cover \( \tilde{\mathcal{E}} \) of \( \mathcal{E} \) is a hyperkähler manifold of dimension \( \dim \tilde{\mathcal{E}} = \dim E = 2(m - 1) \).

Note that, in contrast to the case of strict Enriques varieties, the formula relating index and dimension is not a consequence of the other conditions but is part of the assumptions.

As alluded to above, it is much easier to find examples of strict Enriques stacks compared to strict Enriques varieties. Let \( S \in \text{K3} \) together with a purely non-symplectic automorphism \( f \in \operatorname{Aut}(S) \) of order \( n + 1 \) (which may, and, for \( n + 1 > 2 \), will have fixed points). Then the quotient of the associated Hilbert scheme of points by the induced automorphism \( [X^{[n]}/f^{[n]}] \) is a strict Enriques stack. There are also examples of strict Enriques stacks whose hyperkähler cover is \( K_5(A) \); compare [BNS11, Rem. 4.1].
4.4 Construction of varieties with \( \mathbb{P}^n[k] \)-units

4.4.1 Definition and basic properties

**Definition 4.4.1.** Let \( X \) be a compact Kähler manifold. We say that \( X \) has a \( \mathbb{P}^n[k] \)-unit if \( O_X \) is a \( \mathbb{P}^n[k] \)-object in \( D(X) \). This means that the following two conditions are satisfied

\( (C1) \) The canonical bundle \( \omega_X \) is trivial,
\( (C2) \) There is an isomorphism of \( \mathbb{C} \)-algebras
\[
H^*(O_X) \cong \mathbb{C}[x]/x^{n+1} \quad \text{with} \quad \deg x = k.
\]

**Remark 4.4.2.** If \( X \) has a \( \mathbb{P}^n[k] \)-unit, we have \( \dim X = n \cdot k \). This follows by Serre duality.

**Remark 4.4.3.** For \( n = 1 \), compact Kähler manifolds with \( \mathbb{P}^1[k] \)-units are exactly the strict Calabi–Yau manifolds of dimension \( k \). For \( k = 2 \), compact Kähler manifolds with \( \mathbb{P}^n[2] \)-units are exactly the compact hyperkähler manifolds of dimension \( 2n \); see Observations 4.1.1 and 4.1.2 and Remark 4.2.5.

**Remark 4.4.4.** If \( n \geq 2 \), the number \( k \) must be even. The reason is that the algebra \( H^*(O_X) \) is graded-commutative. Hence, every \( x \in H^k(O_X) \) with \( k \) odd satisfies \( x^2 = 0 \).

Since, in the following, we usually consider the case that \( k > 2 \), we will speak about varieties with \( \mathbb{P}^n[k] \)-units; compare Lemma 4.3.10.

4.4.2 Non-examples

In order to get a better understanding of the notion of varieties with \( \mathbb{P}^n[k] \)-units, it might be instructive to start with some examples which satisfy some of the conditions but fail to satisfy others.

**Products of Calabi–Yau varieties**

Let \( Z \in \text{CY}_8 \) and \( Z' \in \text{CY}_4 \), and set \( X := Z \times Z' \). Then \( \omega_X \) is trivial and by the Künneth formula
\[
H^*(O_X) \cong \mathbb{C}[0] \oplus \mathbb{C}[-4] \oplus \mathbb{C}[-8] \oplus \mathbb{C}[-12].
\]
Hence, as a graded vector space, \( H^*(O_X) \) has the right shape for a \( \mathbb{P}^3[4] \)-unit. As an isomorphism of graded algebras, however, the Künneth formula gives
\[
H^*(O_X) \cong \mathbb{C}[z]/z^2 \otimes \mathbb{C}[z']/(z^2, z') = \mathbb{C}[z,z']/(z^2, z') \quad , \quad \deg z = 8, \deg z' = 4.
\]
This means that, as a \( \mathbb{C} \)-algebra, \( H^*(O_X) \) is not generated in degree 4 so that \( O_X \) is not a \( \mathbb{P}^3[4] \)-object.

**Hilbert schemes of points on Calabi–Yau varieties**

For every smooth projective variety \( X \) and \( n = 2, 3 \), the Hilbert schemes \( X^{[n]} \) of \( n \) points on \( X \) are smooth and projective of dimension \( n \cdot \dim X \). If \( \dim X \geq 3 \) and \( n \geq 4 \), the Hilbert scheme \( X^{[n]} \) is not smooth; see [Che98].

Let now \( X \) be a Calabi–Yau variety of even dimension \( k \) and \( n = 2 \) or \( n = 3 \). Then there is an isomorphism of algebras \( H^*(O_{X^{[n]}}) \cong \mathbb{C}[x]/(x^{n+1}) \) with \( \deg x = k \). The reason is that \( X^{[n]} \) is a resolution of the singularities of the symmetric quotient variety \( X^n/\Sigma_n \), which has
rational singularities, by means of the Hilbert–Chow morphism $X^{[n]} \to X^n/\mathcal{S}_n$. For $k = 2$, the Hilbert scheme of points on a K3 surface is one of the few known examples of a compact hyperkähler manifold which means that $X^{[n]}$ has a $\mathbb{P}^n[2]$-unit for $X \in \text{K3}$. For $\dim X = k > 2$, however, the canonical bundle $\omega_{X^{[n]}}$ is not trivial as this resolution is not crepant.

In contrast, the symmetric quotient stack $[X^n/\mathcal{S}_n]$ has a trivial canonical bundle for $\dim X = k$ an arbitrary even number, and is, in fact, a stack with $\mathbb{P}^n[k]$-unit; see Section 4.6.4 for some further details.

### 4.4.3 Main construction method

In this section, given strict Enriques varieties of index $n + 1$ we construct a series of varieties with $\mathbb{P}^n[2k]$-units. In other words, we prove the implication (iii)$\implies$(ii) of Theorem 4.1.3.

Let $E_1, \ldots, E_k$ be strict Enriques varieties of index $n + 1$. We do not assume that the $E_i$ are non-isomorphic. For the time being, there are known examples of such $E_i$ for $n = 1, 2, 3$; see Theorem 4.3.15. We set $F := E_1 \times \cdots \times E_k$.

**Theorem 4.4.5.** The canonical cover $X := \tilde{F}$ of $F$ has a $\mathbb{P}^n[2k]$-unit.

**Proof.** By definition of the canonical cover, $\omega_X$ is trivial. Hence, Condition (C1) of Definition 4.4.1 is satisfied. It is left to show that $H^*(\mathcal{O}_X) \cong \mathbb{C}[x]/x^{n+1}$ with $\deg x = 2k$. Let $\pi: X \to F$ be the étale cover with $\pi_* \mathcal{O}_X \cong \mathcal{O}_F \oplus \omega_F^{-1} \oplus \cdots \oplus \omega_F^{-n}$. Note that $\omega_F \cong \omega_{E_1} \boxtimes \cdots \boxtimes \omega_{E_k}$. By the Künneth formula together with Lemma 4.3.16, we get

$$H^*(\mathcal{O}_X) \cong H^*(\mathcal{O}_F) \oplus H^*(\omega_F^{-1}) \oplus \cdots \oplus H^*(\omega_F^{-n})$$

$$\cong \left( \bigotimes_{i=1}^k H^*(\mathcal{O}_{E_i}) \right) \oplus \left( \bigotimes_{i=1}^k H^*(\omega_{E_i}^{-1}) \right) \oplus \cdots \oplus \left( \bigotimes_{i=1}^k H^*(\omega_{E_i}^{-n}) \right)$$

$$\cong \mathbb{C} \oplus \mathbb{C} \cdot y_1 \cdots y_k \oplus \cdots \oplus \mathbb{C} \cdot y_1^n \cdots y_k^n$$

$$\cong \mathbb{C}[x]/x^{n+1}$$

where $x := y_1 \cdots y_n$ is of degree $2k$. $\square$

**Remark 4.4.6.** Let $f_i \in \text{Aut}(Y_i)$ be a generator of the group of deck transformations of the cover $Y_i \to E_i$. In other words, $E_i = Y_i/\langle f_i \rangle$. Then we can describe $X$ alternatively as $X = (Y_1 \times \cdots \times Y_k)/G$ where

$$\mu_{n+1}^{k-1} \cong G = \{ f_1^{a_1} \times \cdots \times f_k^{a_k} : a_1 + \cdots + a_k \equiv 0 \mod n + 1 \} \subset \text{Aut}(Y_1 \times \cdots \times Y_n).$$

**Remark 4.4.7.** In the case $n = 1$, one can replace the $Y_i \in \text{K3}$ by strict Calabi–Yau varieties $Z_i$ of dimension $\dim Z_i = d_i$ together with fixed point free involutions $f_i \in \text{Aut}(Z_i)$. Then the same construction gives a variety $X$ with $\mathbb{P}^1[d_1 + \cdots + d_k]$-unit, i.e. a strict Calabi–Yau variety of dimension $\dim X = d_1 + \cdots + d_k$. This coincides with a construction of Calabi–Yau varieties by Cynk and Hulek [CH07].

**Remark 4.4.8.** The construction still works if we replace one of the strict Enriques varieties $E_i$ by an Enriques stack. The reason is that the group $G$ still acts freely on $Y_1 \times \cdots \times Y_k$, even if one of the $f_i$ has fixed points; see Lemma 4.3.6.
4.5 Structure of varieties with $\mathbb{P}^n[k]$-units

4.5.1 General properties

As mentioned in Remark 4.4.3, varieties with $\mathbb{P}^1[k]$-units are exactly the strict Calabi–Yau varieties (not necessarily simply connected) and manifolds with $\mathbb{P}^n[2]$-units are exactly the compact hyperkähler manifolds. Form now on, we will concentrate on the other cases, i.e. we assume that $n > 1$ and $k > 2$. By Remark 4.4.4, this means that $k$ is even.

**Lemma 4.5.1.** Let $X$ be a variety with a $\mathbb{P}^n[k]$-unit. Then there is an étale cover $X' \rightarrow X$ of the form $X' = \prod_i Y_i \times \prod_j Z_j$ with $Y_i \in \text{HK}$ and $Z_j \in \text{CY}$ of even dimension.

**Proof.** Let $X' = T \times Y_i \times \prod_j Z_j$ be a Beauville–Bogomolov cover of $X$ as in Convention 4.3.9. The proof is the same as the first part of the proof of Proposition 4.3.8: We have $\chi(O_X) = n + 1 \neq 0$, hence $\chi(O_{X'}) \neq 0$. It follows that there cannot be a torus or an odd dimensional Calabi–Yau factor occurring in the decomposition on $X'$.

In particular, $X'$ is simply connected, hence agrees with the universal cover:

$$\hat{X} = X' = \prod_i Y_i \times \prod_j Z_j.$$

Since $H^4(O_X) = \mathbb{C}[x]/x^{n+1}$ with $\deg x = k \geq 4$, we see by the Künneth formula that $\hat{X} \rightarrow X$ cannot be an isomorphism; compare (4.1).

**Corollary 4.5.2.** For $X$ a variety with a $\mathbb{P}^n[k]$-unit, $\pi_1(X)$ is a non-trivial finite group.

4.5.2 The case $k = 4$

Now, we focus on the case $k = 4$ where we can determine the decomposition of the universal cover concretely.

**Proposition 4.5.3.** Let $n \geq 3$, and let $X$ be a variety with $\mathbb{P}^n[4]$-unit. Then the universal cover $\hat{X}$ is a product of two hyperkähler varieties of dimension $2n$.

We divide the proof of this statement into several lemmas. So, in the following, let $X$ be a variety with a $\mathbb{P}^n[4]$-unit where $n \geq 3$.

**Lemma 4.5.4.** The universal cover $\hat{X}$ of $X$ is a product of compact hyperkähler manifolds.

**Proof.** By Lemma 4.5.1, we have $\hat{X} = \prod_i Y_i \times \prod_j Z_j$ with $Y_i \in \text{HK}_{2d_i}$ and $Z_j \in \text{CY}_{e_j}$ with $e_j \geq 4$ even. Let $\pi_1(X) \cong G \subset \text{Aut}(\hat{X})$ such that $X = \hat{X}/G$. Analogously to the proof of Proposition 4.3.8, we see that there is an $x \in H^4(O_{\hat{X}})^G \cong H^4(O_X)$ such that $x^n$ is a non-zero multiple of the generator $\prod_i y_i^{d_i} \cdot \prod_j z_j$ of $H^{4n}(O_{\hat{X}})$. In particular, all the $z_j$ have to occur in the expression of $x \in H^4(O_{\hat{X}})$ in terms of the Künneth formula. Hence, $e_j = 4$ for all $j$. We get

$$x = \sum_j z_j + \text{ terms involving the } y_i$$

where we absorb possible non-zero coefficients in the choice of the generators $z_j$ of $H^4(O_{Z_j})$. Both summands of (4.7) are $G$-invariant. This follows by the $G$-invariance of $x$ together with...
Corollary 4.3.2. Hence, one of the two summands must vanish. Consequently, $\hat{X}$ either has no Calabi–Yau or no hyperkähler factors, i.e. $\hat{X} = \prod Y_i$ or $\hat{X} = \prod Z_j$.

Let us assume for a contradiction that the latter is the case. We have $e_j = \dim Z_j = 4$ for all $j$. Since $\dim \hat{X} = \dim X = 4n$, there must be $n$ factors $Z_j \in \text{CY}_4$ of $\hat{X}$. Hence, $\chi(\mathcal{O}_X) = 2^n$. By Lemma 4.3.7, we have

$$\chi(\mathcal{O}_{\hat{X}}) = \chi(\mathcal{O}_X) \cdot \text{ord}(G).$$

(4.8)

Hence, $\chi(\mathcal{O}_X) = n + 1 \mid 2^n$. Furthermore, $G$ must act transitively on $\{z_1, \ldots, z_n\}$. Otherwise, there would be $G$-invariant summands of $x = \sum z_j$ contradicting the assumption that $h^4(\mathcal{O}_X) = 1$. Hence, $n \mid \text{ord } G \mid 2^n$ which, for $n \geq 2$, is not consistent with $n + 1 \mid 2^n$.

Hence, we have $\hat{X} = \prod_{i \in I} Y_i$ with $Y_i \in \text{HK}_{2d}$ for some finite index set $I$ and there is a $G$-invariant

$$0 \neq x = \sum_i c_i y_i^2 + \sum_{i \neq j} c_{ij} y_i y_j \in H^4(\mathcal{O}_{\hat{X}}), \quad c_{ij} \in \mathbb{C}. \quad (4.9)$$

Again by Corollary 4.3.2, both summands in (4.9) are $G$-invariant so that one of them must be zero.

**Lemma 4.5.5.** There is a non-zero $G$-invariant $x \in H^4(\mathcal{O}_{\hat{X}})$ of the form $x = \sum_{i \neq j} c_{ij} y_i y_j$.

**Proof.** Let us assume for a contradiction that we are in the case that $x = \sum y_i^2$ where we hide the coefficients $c_{ij}$ in the choice of the $y_i$. By the same arguments as above, $G$ must act transitively on the set of $y_i$. Hence, by Corollary 4.3.2, we have $\hat{X} = Y^\ell$, $Y \in \text{HK}_{2d}$ with $d\ell = 2n$. We must have $\ell \geq 2$ by Corollary 4.3.5. Then

$$x^2 = \sum_i y_i^4 + 2 \sum_{i \neq j} y_i^2 y_j^2 \in H^8(\mathcal{O}_{\hat{X}})^G$$

and both summands are $G$-invariant. Hence, one of them must be zero and the only possibility for that to happen is that $d < 4$. Since $x^n$ is a scalar multiple of the generator $y_1^d y_2^d \cdots y_\ell^d$ of $H^{4n}(\mathcal{O}_{\hat{X}})$, we must have $d = 2$. Hence, $\ell = n$ and $\chi(\mathcal{O}_{\hat{X}}) = 3^n$. By (4.8) and the fact that $G$ acts transitively on $\{y_1, \ldots, y_n\}$, we get the contradiction $n \mid 3^n$ and $n + 1 \mid 3^n$.

**Lemma 4.5.6.** We have $|I| = 2$ which means that $\hat{X} = Y \times Y'$ with $Y, Y' \in \text{HK}$.

**Proof.** Let $0 \neq x = \sum_{i \neq j} c_{ij} y_i y_j \in H^4(\mathcal{O}_{\hat{X}})^G$ with $c_{ij} \in \mathbb{C}$, some of which might be zero, as in Lemma 4.5.5. As already noted above, we have $|I| \geq 2$ by Corollary 4.3.5. Let us assume that $|I| \geq 3$. This assumption will be divided into several subcases, each of which leads to a contradiction. We have

$$x^2 = \sum_{i \neq j} c_{ij} y_i^2 y_j^2 + \sum_{h \neq i \neq j} c_{hi} c_{ij} y_h y_i y_j + \sum_{g \neq h \neq i \neq j} \hat{c}_{ghi} y_g y_h y_i y_j, \quad \hat{c}_{ghi} = c_{gh} c_{ij} + \ldots$$

(4.10)

All three summands are $G$-invariant by Corollary 4.3.2, hence two of them must be zero. For one of the first two summands of (4.10) to be zero, the square of some $y_i$ must be zero, i.e. some $Y_{i_0}$ must be a K3 surface. Write the index set $I$ of the decomposition $\hat{X} = \prod_{i \in I} Y_i$ as $I = N \cup M$ where $N = G \cdot i_0$ is the orbit of $i_0$. Here we consider the $G$-action on $I$ given...
by the permutation part of the autoequivalences in \( G \subset \text{Aut}(\hat{X}) \); see Lemma 4.3.1. With this notation, \( Y_j = Y_{ij} \in K3 \) for \( j \in N \).

Let us first consider the case that \( G \) acts transitively on the factors of the decomposition of \( \hat{X} \), i.e. \( I = N \). Then, by dimension reasons, \( |I| = 2n \). In other words, \( \hat{X} = Y_{2n} \) with \( Y \in K3 \). Hence, \( \chi(O_{\hat{X}}) = 2^{2n} \). By (4.8) we get the contradiction \( 2n | 2^{2n} \) and \( n + 1 | 2^{2n} \).

In the case that \( M \neq 0 \), all the non-zero coefficients \( c_{ij} \) in the \( G \)-invariant \( x = \sum_{i \neq j} c_{ij} y_i y_j \) must be of the form \( i \in N \) and \( j \in M \) (or the other way around). Indeed, otherwise we would have \( G \)-invariant proper summands of \( x \) in contradiction to the assumption \( H^4(O_{\hat{X}})^G = \langle x \rangle \).

Furthermore, for all \( i \in N \) there must be a non-zero \( c_{ii'} \) and for all \( j' \in M \) there must be a non-zero \( c_{jj'} \) since \( x^n \) is a non-zero multiple of the generator \( \prod_{i \in N} y_i \cdot \prod_{j \in M} y_{j'} \) of \( H^4(O_{\hat{X}}) \). Hence, to avoid proper \( G \)-invariant summands of \( x \), the group \( G \) must also act transitively on \( M \). It follows that \( \hat{X} = Y^\ell \times (Y')^{\ell'} \) where \( \ell = |N| \), \( \ell' = |M| \), \( Y \in K3 \), and \( Y' \in \text{HK}_{2d}^e \) for some \( \ell' \). Now, \( x^n \) is a non-zero multiple of

\[
\prod_{i=1}^{\ell} y_i \cdot \prod_{j=1}^{\ell'} (y_{j'})^{d'} \in H^4n(O_{\hat{X}}).
\]

Since all the non-zero summands of \( x \) are of the form \( c_{ij} y_i y_{j'} \), we get that \( \ell = n = \ell' \cdot d' \). In particular,

\[
\hat{X} = Y^n \times (Y')^{\ell'}.
\] (4.11)

First, we consider for a contradiction the case that \( \ell' = 1 \), hence \( \hat{X} = Y^n \times Y' \) with \( Y \in K3 \) and \( Y' \in \text{HK}_{2n} \). Then, by (4.8), we get \( \text{ord } G = 2^n \). We have (up to coefficients which we avoid by the correct choice of the \( y_i \)), \( x = \sum_{i=1}^{n} y_i y_i' \). Accordingly, \( x^2 = \sum_{i \neq j} y_i y_j (y_j')^2 \). Hence, \( G \) acts transitively on \( \{y_1, \ldots, y_n\} \) as well as on \( \{y_i y_j \mid 1 \leq i < j \leq n\} \). We get the contradiction \( n | 2^n \) and \( (\frac{n}{2}) | 2^n \).

Note that, for this to be a contradiction, we need the assumption \( n \geq 3 \). Indeed, in Section 4.6.2, we will see examples of a variety \( X \) with a \( \mathbb{P}^2[4] \)-unit whose canonical covers are of the form \( \hat{X} = Y^2 \times Y' \) with \( Y \in K3 \) and \( Y' \in \text{HK}_{4} \).

Now, let \( \ell' > 1 \) in (4.11). Then, we get

\[
x^2 = \sum_{i \neq j, i' \neq j'} c_{ij} c_{ij'} y_i y_j (y_{j'})^2 + \sum_{i \neq j, i' \neq j'} \tilde{c}_{ijj'} y_i y_{j} y_{j'} (y_{j'})^2, \quad \tilde{c}_{ijj'} = c_{ij} c_{ij'} + c_{ij'} c_{ij} \quad (4.12)
\]

where both summands are \( G \)-invariant. Hence, in order to avoid linearly independent classes in \( H^8(O_{\hat{X}})^G \), one of them must be zero.

Let us assume for a contradiction that all the \( \tilde{c}_{ijj'} \) are zero. Then all the \( c_{ij} \) with \( i \in N \) and \( i' \in M \) are non-zero. Indeed, as mentioned above, given \( i \in N \) and \( i' \in M \), there exist \( j \in N \) and \( j' \in M \) such that \( c_{ij} \neq 0 \neq c_{ij'} \). By \( \tilde{c}_{ijj'} = 0 \), it follows that also \( c_{ij} \neq 0 \neq c_{ij} \).

Given pairwise distinct \( h, i, j \in N \) and \( i', j' \in M \) we consider the following term, which is the coefficient of \( y_h y_i y_j (y_{j'})^2 y_{j'}^2 \) in \( x^3 \),

\[
C := c_{hi} c_{i} c_{ij} + c_{hi} c_{ij} c_{ij'} + c_{hj} c_{ij'} c_{ji'} + c_{hj} c_{i} c_{ji'} \quad (4.13)
\]

\[
= c_{hi} \tilde{c}_{ijj'} + c_{hj} \tilde{c}_{ijj'}
= c_{hi} \tilde{c}_{hi} c_{j} + c_{hj} \tilde{c}_{hi} c_{j'}
= c_{ji} \tilde{c}_{hi} c_{j'} + c_{hi} \tilde{c}_{hi} c_{j'} c_{j'}.
\]
By the vanishing of the $\hat{c}$, we get
$$C = c_{hi'}c_{ii'}c_{jj'} = c_{hi'}c_{ij'}c_{jj'} = c_{hi'}c_{ii'}c_{jj'}.$$ 
By the non-vanishing of all the $c$, we get $C \neq 0$. But, at the same time, by (4.13), we have $3C = C$, a contradiction.

We conclude that the first summand of (4.12) is zero. This can only happen for $(y'_r)^2 = 0$, hence $Y' \in K3$. Then $\chi(O_{\hat{X}}) = 2^{2n}$ and, as before, we get the contradiction that $n \mid 2^{2n}$ and $n + 1 \mid 2^{2n}$. □

**Proof of Proposition 4.5.3.** By now, we know that $\hat{X} = Y \times Y'$ with $Y \in HK_{2d}$ and $Y' \in HK_{2d'}$, and $x = yy'$. We have $d + d' = 2n$. Furthermore, $0 \neq x^n = y^n(y')^n$. Hence, $d = n = d'$. □

**Remark 4.5.7.** The proof of Proposition 4.5.3 becomes considerably simpler if one assumes that $n+1$ is a prime number. In this case, it follows directly by Lemma 4.3.7 that the universal cover must have a factor $Y \in HK_{2n}$. Hence, there are much fewer cases one has to deal with.

**Theorem 4.5.8.** Let $n \geq 3$, and let $X$ be a variety with a $\mathbb{P}^n[4]$-unit.

1. We have $X = (Y \times Y')/G$ with $Y, Y' \in HK_{2n}$. The group $\pi_1(X) \cong G \subset \text{Aut}(Y \times Y')$ acts freely, and is of the form $G = \langle f \times f' \rangle$ with $f \in \text{Aut}(Y)$ and $f' \in \text{Aut}(Y')$ purely symplectic of order $n + 1$.

2. If $n + 1 = p^d$ is a prime power, at least one of the cyclic groups $\langle f \rangle \subset \text{Aut}(Y)$ and $\langle f' \rangle \subset \text{Aut}(Y')$ acts freely.

Before giving the proof of the theorem, let us restate, for convenience, the special case of Lemma 4.3.6 for automorphisms of products with two factors.

**Lemma 4.5.9.** Let $X$ and $Y$ be manifolds, $g, f \in \text{Aut}(X)$ and $h \in \text{Aut}(Y)$.

1. $g \times h \in \text{Aut}(X \times Y)$ is fixed point free if and only if at least one of $g$ and $h$ is fixed point free.

2. Let $\varphi := (f \times g) \circ (1 2) \in \text{Aut}(X^2)$ be given by $(a, b) \mapsto (f(b), g(a))$. Then, $\varphi$ is fixed point free if and only if $f \circ g$ and $g \circ f$ are fixed-point free.

**Proof of Theorem 4.5.8.** The fact that $X = (Y \times Y')/G$ with $Y, Y' \in HK_{2n}$ and $G \cong \pi_1(X)$ is just a reformulation of Proposition 4.5.3. By the proof of this proposition, we see that $H^*(O_{Y \times Y'})^G \cong H^*(O_X)$ is generated by $x = yy'$ in degree 4.

Let us assume for a contradiction that $G$ contains an element which permutes the factors $Y$ and $Y'$, in which case we have $Y = Y'$ by Lemma 4.3.1. In other words, there exists an $\varphi = (f \times g) \circ (1 2) \in G$ as in Lemma 4.5.9 2. Hence, $f \circ g$ is fixed-point free. By Lemma 4.3.4, the composition $f \circ g$ is non-symplectic, i.e. $q_{fog} \neq 1$. But $q_{fog} = q_f \cdot q_g$ so that $\varphi$ acts non-trivially on $x = yy'$ in contradiction to the $G$-invariance of $x$.

Hence, every element of $G$ is of the form $g \times h$ as in Lemma 4.3.6 1. We consider the group homomorphisms $\varrho_Y : G \rightarrow \mathbb{C}^*$ and $\varrho_{Y'} : G \rightarrow \mathbb{C}^*$. Their images are of the form $\mu_m$ and $\mu_{m'}$ respectively. We must have $m, m' \geq n + 1$. Indeed, $y^m$ and $(y')^{m'}$ are $G$-invariant but, for $m \leq n$ or $m' \leq n$, not contained in the algebra generated by $x = yy'$. Since $|G| = n + 1$, assertion 1 follows.
Let now $n + 1 = p^r$ be a prime power and $G = \langle f \rangle$. Let us assume for a contradiction that there exist $a, b \in \mathbb{N}$ with $n + 1 = p^r + a, b$ such that $f^a$ and $(f')^b$ have fixed points. Note that, in general, if an automorphism $g$ has fixed-points, also all of its powers have fixed points. Furthermore, for two elements $a, b \in \mathbb{Z}/(p^r)$ we have $a \in \langle b \rangle$ or $b \in \langle a \rangle$. Hence, $(f \times f')^a$ or $(f \times f')^b$ has fixed points in contradiction to part 1.

This proves the implication (i) $\implies$ (iii) of Theorem 4.1.3. Indeed, for $n + 1$ a prime power, the above Theorem says that $Y/\langle f \rangle$ or $Y'/\langle f' \rangle$ is a strict Enriques variety; see Proposition 4.6.1. Note that Theorem 4.5.8 above does not hold for $n = 2$; see Section 4.6.2. However, both conditions (i) and (iii) of Theorem 4.1.3 hold true for $n = 2$; see Theorem 4.3.15 and Corollary 4.1.4.

Remark 4.5.10. The proof of part 2 of Theorem 4.5.8 does not work if $n + 1$ is not a prime power. For example, if $n + 1 = 6$, one could obtain a variety with $\mathbb{P}^5[4]$-unit as a quotient $X = (Y \times Y')/(f \times g)$ with $Y, Y' \in \text{HK}_{10}$ such that $f$ and $g$ are purely non-symplectic of order 6, and $f, f^2, f^4, f^5, g, g^3, g^5$ are fixed point free but $f^3, g^2, g^4$ are not. The author does not know whether hyperkähler manifolds together with these kinds of automorphisms exist.

4.6 Further remarks

4.6.1 Further constructions using strict Enriques varieties

Given strict Enriques varieties of index $n + 1$, there are, for $k \geq 6$, further constructions of varieties with $\mathbb{P}^n[k]$-units besides the one of Section 4.4.3. Let $Y \in \text{HK}_{2n}$ and $f \in \text{Aut}(Y)$ purely symplectic of order $n + 1$ such that $\langle f \rangle$ acts freely, i.e. the quotient $E = Y/\langle f \rangle$ is a strict Enriques variety. We consider the $(n + 1)$-cycle $\sigma := (1 \, 2 \, \ldots \, n + 1) \in S_{n+1}$ and the subgroup $G(Y) \subset \text{Aut}(Y^{n+1})$ given by

$$G(Y) := \{ (f^{a_1} \times \cdots \times f^{a_{n+1}}) \circ \sigma^a \mid a_1 + \cdots + a_{n+1} \equiv a \mod n + 1 \}.$$ 

Every non-trivial element of $G(Y)$ acts without fixed points on $Y^{n+1}$ by Lemma 4.3.6. There is the surjective homomorphism

$$G(Y) \to \mathbb{Z}/(n + 1)\mathbb{Z}, \quad (f^{a_1} \times \cdots \times f^{a_{n+1}}) \circ \sigma^a \mapsto a \mod n + 1$$

and we denote the fibres of this homomorphism by $G_a(Y)$.

Now, consider further $Z_1, \ldots, Z_k \in \text{HK}_{2n}$ together with purely non-symplectic $g_i \in \text{Aut}(Z_i)$ of order $n + 1$ such that $\langle g_i \rangle$ acts freely and

$$g Z_i g_i = g \gamma f \quad \text{for all } i = 1, \ldots, k.$$ (4.14)

The equality (4.14) can be achieved as soon as we have any purely non-symplectic automorphisms $g_i \in \text{Aut}(Z_i)$ of order $n + 1$ by replacing the $g_i$ by an appropriate power $g_i^\nu$ with $\gcd(\nu, n + 1) = 1$. We consider the subgroup $G(Y; Z_1, \ldots, Z_k) \subset \text{Aut}(Y^{n+1} \times Z_1 \times \cdots \times Z_k)$ given by

$$G(Y; Z_1, \ldots, Z_k) := \{ F \times g_1^{b_1} \times \cdots \times g_k^{b_k} \mid F \in G_a(Y), a + b_1 + \cdots + b_k \equiv 0 \mod n + 1 \}.$$ 

Proposition 4.6.1. The quotient $X := (Y^{n+1} \times Z_1 \times \cdots \times Z_k)/G(Y; Z_1, \ldots, Z_k)$ is a smooth projective variety with $\mathbb{P}^n[2(n + 1 + k)]$-unit.
Proof. One can check using Lemma 4.3.6 that the group \( G := G(Y; Z_1, \ldots, Z_k) \) acts freely on \( X' := Y^{n+1} \times Z_1 \times \cdots \times Z_k \). Hence, \( X \) is indeed smooth.

By the defining property of the elements of \( G(Y; Z_1, \ldots, Z_k) \) together with (4.14), we see that \( x := y_1 y_2 \cdots y_{n+1} z_1 z_2 \cdots z_k \) is \( G \)-invariant. Hence, as \( x' \neq 0 \) for \( 0 \leq i \leq n \), we get the inclusion

\[
\mathbb{C}[x]/x^{n+1} \subset H^*(O_{X'})^G \cong H^*(O_X) , \quad \operatorname{deg} x = 2(n + 1 + k).
\] (4.15)

Also, \( \operatorname{ord} G(Z_1, \ldots, Z_k) = (n + 1)^{n+1+k-1} \). By Lemma 4.3.7, we get \( \chi(O_{X'}) = n + 1 \) so that the inclusion (4.15) must be an equality which is (C2). Finally, the canonical bundle \( \omega_X \) is trivial since \( G \) acts trivially on \( \langle x^n \rangle = H^{\dim X'}(O_{X'}) \cong H^0(\omega_{X'}) \). \( \square \)

Remark 4.6.2. For \( n \geq 2 \), the group \( G(Y; Z_1, \ldots, Z_k) \) is not abelian. Since \( X' \to X \) is the universal cover, we see that, for \( k \geq 4 \), there are examples of varieties with \( \mathbb{P}^n[k] \)-units which have a non-abelian fundamental group.

Remark 4.6.3. Again, for one \( i \in \{1, \ldots, k\} \) we may drop the assumption that \( \langle g_i \rangle \) acts freely; compare Remark 4.4.8.

Remark 4.6.4. One can further generalise the above construction as follows. Consider hyperkähler manifolds \( Y_1, \ldots, Y_m, Z_1, \ldots, Z_k \in \mathbb{H}K_{2n} \) together with \( f_i \in \operatorname{Aut}(Y_i) \) and \( g_j \in \operatorname{Aut}(Z_j) \) purely non-symplectic of order \( n + 1 \) such that the generated cyclic groups act freely. Set \( X' := Y_1^{n+1} \times \cdots \times Y_m^{n+1} \times Z_1 \times \cdots \times Z_k \) and consider \( G := G(Y_1, \ldots, Y_m; Z_1, \ldots, Z_k) \subset \operatorname{Aut}(X') \) given by

\[
G = \{ F_1 \times \cdots \times F_m \times g_1^{b_1} \times \cdots \times g_k^{b_k} \mid F_i \in G_{a_i}(Y) , a_1 + \cdots + a_m + b_1 + \cdots + b_k \equiv 0 \mod n+1 \} .
\]

Then, \( X := X'/G \) has a \( \mathbb{P}^n[2(m(n+1)+k)] \)-unit.

Remark 4.6.5. In the case \( n = 1 \), one may replace the K3 surfaces \( Y_i \) and \( Z_j \) by strict Calabi–Yau varieties of arbitrary dimensions. Still, the quotient \( X \) will be a strict Calabi–Yau variety.

4.6.2 A construction not involving strict Enriques varieties

As mentioned in Section 4.5.2, there is a variety \( X \) with \( \mathbb{P}^2[4] \)-unit whose universal cover \( \tilde{X} \) is not a product of two hyperkähler varieties of dimension 4. This shows that the assumption \( n \geq 3 \) in Proposition 4.5.3 is really necessary.

For the construction, let \( Z \) be a strict Calabi–Yau variety of dimension \( \dim Z = e \) together with a fixed point free involution \( \iota \in \operatorname{Aut}(Z) \). Necessarily, \( g_{Z, \iota} = -1 \); see Lemma 4.3.4. Furthermore, let \( Y \in \mathbb{H}K_4 \) together with a purely non-symplectic \( f \in \operatorname{Aut}(Y) \) of order 4. Note that \( g \) must have fixed points on \( Y \). Such pairs \( (Y, f) \) exist. Take a K3 surface \( S \) (an abelian surface \( A \)) together with a purely non-symplectic automorphism of order 4 and \( Y = S^{[2]} \) \( (Y = K_2 A) \) together with the induced automorphism.

Now, consider \( G(Z) \subset \operatorname{Aut}(Z^2) \) as in the previous section. It is a cyclic group of order 4 with generator \( g = (\iota \times \operatorname{id}) \circ (1, 2) \). Set \( X' = Y \times Z^2 \) and \( G := \langle f \times g \rangle \subset \operatorname{Aut}(X') \). The group \( G \) acts freely, since \( G(Z) \) does; see Lemma 4.5.9. One can check that \( x = y_{21} + iy_{22} \in H^{2+e}(O_{X'}) \) is \( G \)-invariant. By the same argument as in the proof of Proposition 4.6.1, we conclude that \( X \) has a \( \mathbb{P}^2[2+e] \)-unit. In particular, in the case that \( Z \in \mathbb{K}3 \), we get a variety with \( \mathbb{P}^2[4] \)-unit.
4.6.3 Possible construction for $k = 6$

In contrast to the case $k = 4$ and $n + 1$ a prime power (see Theorem 4.1.3), there might be a variety with $\mathbb{P}^n[6]$-unit even if there is no Enriques variety of index $n + 1$ but one of index $2n + 1$. Of course, since there are at the moment only known examples of strict Enriques varieties of index 2, 3, and 4, this is only hypothetical.

Indeed, let $Y \in \text{HK}_{4n}$ together with subgroup $\langle f \rangle \subset \text{Aut}(Y)$ acting freely, where $f$ is purely non-symplectic of order $2n + 1$ and let $Y' \in \text{HK}_{2n}$ together with $f' \in \text{Aut}(Y')$ non-symplectic of order $n + 1$ with $\varrho_{Y,f} = \varrho_{Y',f'}$. Necessarily, $f'$ has fixed points; see Lemma 4.3.4. Then $G = \langle f \times f'^2 \rangle$ acts freely on $Y$ and $x = y^2 \cdot y'$ is $G$-invariant. It follows that $X = (Y \times Y')/G$ has a $\mathbb{P}^n[6]$-unit.

4.6.4 Stacks with $\mathbb{P}^n[k]$-units

Let $\mathcal{X}$ be a smooth projective stack. In complete analogy to the case of varieties, we say that $\mathcal{X}$ has a $\mathbb{P}^n[k]$-unit if $\mathcal{O}_X \in D(\mathcal{X})$ is a $\mathbb{P}^n[k]$-object. Again, this means that:

(C1') The canonical bundle $\omega_\mathcal{X}$ is trivial,

(C2') There is an isomorphism of $\mathbb{C}$-algebras $H^*(\mathcal{O}_\mathcal{X}) \cong \mathbb{C}[x]/x^{n+1}$ with $\deg x = k$.

In contrast to the case of varieties, it is very easy to construct stacks with $\mathbb{P}^n[k]$-units.

Let $Z \in \text{CY}_k$ with $k$ even. Then, the symmetric group $\mathfrak{S}_n$ acts on $Z^n$ by permutation of the factors and we call the associated quotient stack $\mathcal{X} = [Z^n/\mathfrak{S}_n]$ the symmetric quotient stack. Then, as $k = \dim Z$ is even, the canonical bundle of $\mathcal{X}$ is trivial; see [1, Sect. 5.4]. Condition (C2') follows by the Künneth formula

$$H^*(\mathcal{O}_\mathcal{X}) \cong H^*(\mathcal{O}_Z)^{\mathfrak{S}_n} \cong (H^*(\mathcal{O}_Z)^{\otimes n})^{\mathfrak{S}_n} \cong S^n(H^*(\mathcal{O}_Z)).$$

There are also plenty of other examples of stacks with $\mathbb{P}^n[k]$-units. Let $S \in \text{K3}$ with $\iota \in S$ a non-symplectic involution and $\iota^{[n]} \in \text{Aut}(S^{[n]})$ the induced automorphism on the Hilbert scheme of $n$ points on $S$. Then, for $n$ even, the associated quotient stack $[X^{[n]}/\iota^{[n]}]$ has a $\mathbb{P}^{n/2}[4]$-unit. In contrast, if $\iota$ is fixed point free and $n$ is odd, $\iota^{[n]}$ is again fixed point free and the quotient $X^{[n]}/\iota^{[n]}$ is an OS Enriques variety; see [OS11, Prop. 4.1].

Also, all the constructions of the earlier sections lead to stacks with $\mathbb{P}^n[k]$-units if we replace the strict Enriques varieties by strict Enriques stacks.

4.6.5 Derived invariance of strict Enriques varieties

In [Abu15], Abuaf conjectured that the homological unit is a derived invariant of smooth projective varieties. This means that for two varieties $X_1, X_2$ with $D(X_1) \cong D(X_2)$ we should have an isomorphisms of $\mathbb{C}$-algebras $H^*(\mathcal{O}_{X_1}) \cong H^*(\mathcal{O}_{X_2})$.

In regard to this conjecture, one would like to prove that the class of varieties with $\mathbb{P}^n[k]$-units is stable under derived equivalences. This is true for $k = 2$: In [HN11], it is shown that the class of compact hyperkähler manifolds is stable under derived equivalence. However, the methods of the proof do not seem to generalise to higher $k$. At least, we can use the result of [HN11] in order to show that the class of strict Enriques varieties is derived stable.

**Lemma 4.6.6.** Let $E_1$ be a strict Enriques variety of index $n + 1$ and $E_2$ a Fourier–Mukai partner of $E_2$, i.e. $E_2$ is a smooth projective variety with $D(E_1) \cong D(E_2)$. Then $E_2$ is also a strict Enriques variety of the same index $n + 1$. 

126
Proof. By Proposition 4.3.14, condition (S1) of a strict Enriques variety of index $n + 1$ can be replaced by the condition $\dim E_1 = 2n$. The dimension of a variety and the order of its canonical bundle are derived invariants; see e.g. [Huy06, Prop. 4.1]. Hence, also $\dim E_2 = 2n$ and $\text{ord } \omega_{E_2} = n + 1$.

It remains to show that the canonical cover $\widetilde{E}_2$ is again hyperkähler. Indeed, the equivalence $D(E_1) \cong D(E_2)$ lifts to an equivalence of the canonical covers $D(\widetilde{E}_1) \cong D(\widetilde{E}_2)$ and the class of hyperkähler varieties is stable under derived equivalences; see [BM98] and [HN11], respectively.

4.6.6 Autoequivalences of varieties with $\mathbb{P}^n[k]$-unit

As mentioned in Remark 4.2.7, every $\mathbb{P}^n[k]$-object $E \in D(X)$ induces an autoequivalence, called $\mathbb{P}$-twist, $P_E \in \text{Aut}(D(X))$. This can be seen as a special case of [Add16, Thm. 3] or as a straight-forward generalisation of [HT06, Prop. 2.6]. We will describe the twist only in the special case $E = \mathcal{O}_X$. In particular, we assume that $X$ has a $\mathbb{P}^n[k]$-unit. Then, by Remark 4.2.8, every line bundle $L \in \text{Pic} X$ is a $\mathbb{P}^n[k]$-object too. However, it suffices to understand the twist $P_X := P_{\mathcal{O}_X}$ as we have $P_L = M_L P_X M_L^{-1}$ where $M_L = (\underline{\text{r}}) \otimes L$ is the autoequivalence given by tensor product with $L$; see [6, Lem. 2.4].

The $\mathbb{P}$-twist along $\mathcal{O}_X$ is constructed as the Fourier–Mukai transform $P_X := \text{FM}_\mathcal{O} : D(X) \to D(X)$ where

$$Q = \text{cone}(\text{cone}(\mathcal{O}_{X \times X} \xrightarrow{x \otimes \text{id} - \text{id} \otimes x} \mathcal{O}_{X \times X}) \xrightarrow{r} \mathcal{O}_\Delta) \in D(X \times X).$$

Here, $x$ is a generator of $H^k(\mathcal{O}_X) \cong \text{Hom}(\mathcal{O}_X[-k], \mathcal{O}_X)$ and $r : \mathcal{O}_{X \times X} \to \mathcal{O}_\Delta$ is the restriction of sections to the diagonal. The double cone makes sense, since $r \circ (x \otimes \text{id} - \text{id} \otimes x) = 0$; see [HT06, Sect. 2] for details. On the level of objects $F \in D(X)$, the twist $P_X$ is given by

$$P_X(F) = \text{cone}\left(\text{cone}(H^*(F) \otimes \mathcal{O}_X[-k] \to H^*(F) \otimes \mathcal{O}_X) \to F\right). \quad (4.16)$$

We summarise the main properties of the twist $P_X$ in the following

Proposition 4.6.7. The $\mathbb{P}$-twist $P_X : D(X) \to D(X)$ is an autoequivalence with the properties

1. $P_X(\mathcal{O}_X) = \mathcal{O}_X[-k(n + 1) + 2],$

2. $P_X(F) = F$ for $F \in \mathcal{O}_X^\perp = \{ F \in D(X) \mid \text{Hom}^*(\mathcal{O}_X, F) = 0\},$

3. Let $\Phi \in \text{Aut}(D(X))$ with $\Phi(\mathcal{O}_X) = \mathcal{O}_X[m]$ for some $m \in \mathbb{Z}$. Then the autoequivalences $\Phi$ and $P_X$ commute.

Proof. For the first two properties, see [HT06, Sect. 2] or [Add16, Sect. 3.4&3.5]. Part 3 follows from [6, Lem. 2.4].

Lemma 4.6.8. Let $X$ be a variety with $\mathbb{P}^n[k]$-unit with $k \geq 2$ (not an elliptic curve). Let $Z_1, Z_2 \subset X$ be two disjoint closed subvarieties and set $F := R^\text{Hom}(P_X(\mathcal{O}_{Z_1}), P_X(\mathcal{O}_{Z_2}))$. Then $\text{Hom}^*(\mathcal{O}_X, F) = H^*(F) = 0$ and $F \neq 0$. In particular, the orthogonal complement of $\mathcal{O}_X$ is non-trivial.
Proof. Clearly, $\text{Hom}^*(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}) = 0$. Using the fact that the equivalence $P_X$ is, in particular, fully faithful and standard compatibilities between derived functors, we get

$$0 = \text{Hom}^*(P_X(\mathcal{O}_{Z_1}), P_X(\mathcal{O}_{Z_2})) = \text{Hom}^*(\mathcal{O}_X, R\mathcal{H}om(P_X(\mathcal{O}_{Z_1}), P_X(\mathcal{O}_{Z_2}))).$$

It is left to show that $F := R\mathcal{H}om(P_X(\mathcal{O}_{Z_1}), P_X(\mathcal{O}_{Z_2})) \neq 0$. We denote by $\alpha_i$ the top non-zero degree of $H^i(\mathcal{O}_{Z_i})$ for $i = 1, 2$. Let $V := X \setminus (Z_1 \cup Z_2)$. Then by (4.16), the cohomology of $P_X(\mathcal{O}_{Z_i})$ is concentrated in degrees between $-1$ and $\alpha_i + k - 2$ with $H^{-1}(P_X(\mathcal{O}_{Z_i}))|_V \cong \mathcal{O}_V$ and $H^{\alpha_i + k - 2}(P_X(\mathcal{O}_{Z_i}))|_V \cong \mathcal{O}_V \otimes H^{\alpha_i}(\mathcal{O}_{Z_i})$. Hence, the spectral sequence

$$E_2^{p,q} = \oplus_i \text{Ext}^p(H^i(P(\mathcal{O}_{Z_1})), H^{i+q}(P(\mathcal{O}_{Z_1})))|_V \quad \implies \quad E_2^{p+q} = H^{p+q}(F)|_V$$

is concentrated in the quadrant to the upper right of $(0, -\alpha_1 - k + 1)$. Furthermore, we have $E_2^{0, -\alpha_1 - k + 1} \cong \mathcal{O}_V \otimes H^{\alpha_1}(\mathcal{O}_{Z_1}) \neq 0$. Hence $H^{-\alpha_1 - k + 1}(F) \neq 0$. \hfill $\Box$

Let now $X$ be obtained from strict Enriques varieties via the construction of Section 4.4.3. This means that $X = (Y_1 \times \cdots \times Y_k)/G$ with $Y_i \in \mathbb{H}K_{2n}$ and

$$G = \{ f_1a_1 \times \cdots \times f_ka_k \mid a_1 + \cdots + a_k \equiv 0 \pmod{n + 1} \}$$

where the $f_i \in \text{Aut}(Y_i)$ are purely non-symplectic of order $n + 1$. There are the $\mathbb{P}$-twists $P_{Y_i} := P_{\mathcal{O}_{Y_i}} \in \text{Aut}(D(Y_i))$ whose Fourier–Mukai kernels we denote by $Q_i$. These induce autoequivalences $P_{Y_i}' := \mathcal{F}\mathcal{M}_{Q_i} \in \text{Aut}(D(Y_1 \times \cdots \times Y_k))$ where

$$Q_i' = \mathcal{O}_{\Delta Y_1} \boxtimes \cdots \boxtimes Q_i \boxtimes \cdots \boxtimes \mathcal{O}_{\Delta Y_k} \in D((Y_1 \times Y_1) \times \cdots \times (Y_1 \times Y_i) \times \cdots (Y_k \times Y_k)).$$

We have

$$P_{Y_i}'(F_1 \boxtimes \cdots \boxtimes F_k) = F_1 \boxtimes \cdots \boxtimes P_{Y_i}(F_i) \boxtimes \cdots \boxtimes F_k. \quad (4.17)$$

We will use in the following the identification $D(X) \cong D_G(X')$ of the derived category of $X$ with the derived category of $G$-linearised coherent sheaves on the cover $X' = Y_1 \times \cdots \times Y_k$; see e.g. [BKR01, Sect. 4] or [1] for details. One can check that the $Q_i$ are $(f_i)$-linearisable, hence the $Q_i'$ are $G$-linearisable. It follows that the autoequivalences $P_{Y_i}'$ descend to autoequivalences $\tilde{P}_{Y_i} \in \text{Aut}(D_G(X')) \cong \text{Aut}(D(X))$; see [1, Thm. 1.1]. One might expect that the composition of the $\tilde{P}_{Y_i}$ equals $P_X$ but this is not the case.

**Proposition 4.6.9.** There is an injective group homomorphism $\mathbb{Z}^{\oplus k+2} \hookrightarrow \text{Aut}(D(X))$ given by

$$e_{k+1} \mapsto P_X, \quad e_{k+2} \mapsto [1], \quad e_i \mapsto \tilde{P}_{Y_i} \quad \text{for } i = 1, \ldots, k.$$

**Proof.** Under the equivalence $D(X) \cong D_G(X')$, the structure sheaf $\mathcal{O}_X \in D(X)$ corresponds to $\mathcal{O}_X = \mathcal{O}_{Y_1} \boxtimes \cdots \boxtimes \mathcal{O}_{Y_k}$ equipped with the natural linearisation. By (4.17) and Proposition 4.6.7(1), we get

$$\tilde{P}_{Y_i}(\mathcal{O}_X) \cong \mathcal{O}_{Y_1} \boxtimes \cdots \boxtimes (\mathcal{O}_{Y_i}[2n]) \boxtimes \cdots \boxtimes \mathcal{O}_{Y_k} \cong \mathcal{O}_X[-2n].$$

Hence, by 4.6.7(3), the $\tilde{P}_{Y_i}$ commute with $P_X$. By a similar argument, one can see that the $\tilde{P}_i$ commute with one another. The shift functor $[1]$ commutes with every autoequivalence of the triangulated category $D(X)$. In summary, we have shown by now that the homomorphism $\mathbb{Z}^{\oplus k+2} \hookrightarrow \text{Aut}(D(X))$ is well-defined.
For the injectivity, let us fix for every $i = 1, \ldots, n$ a $G$-linearisable $F_i \in \mathcal{O}_Y^Z$. For example, let $Z_1$ and $Z_2$ in Lemma 4.6.8 be two different $\langle f_i \rangle$-orbits in $Y_i$. Let $a_1, \ldots, a_k, b, c \in Z$ and set $\Psi := P^b_{Y_i} \circ \cdots \circ P^{a_k}_{Y_i} \circ P^c_X \langle c \rangle$. By plugging various box-products of the $\mathcal{O}_{Y_i}$ and $F_i$ into $\Psi$ we can show that $\Psi \cong \text{id}$ implies $0 = a_1 = a_2 = \cdots = a_k = b = c$; this is very similar to computations done in [Add16, Sect. 1.4] or the proof of [5, Prop. 3.18].

\textbf{Remark 4.6.10.} In the known examples, the $Y_i$ are generalised Kummer varieties; compare Section 4.3.4. In these cases, there are many more $\mathbb{P}$-objects in $D(Y_i)$ which induce further autoequivalences on $X$; see [6, Sect. 6].

\textbf{Corollary 4.6.11.} Let $X$ be a variety with $\mathbb{P}^n[4]$-unit for $n \geq 3$. Then, there is an embedding $\mathbb{Z}^4 \subset \text{Aut}(D(X))$.

\textit{Proof.} By Theorem 4.5.8, we are in the situation of the above proposition. \hfill \square

\textbf{4.6.7 Varieties with $\mathbb{P}^n[k]$-units as moduli spaces}

All the examples of varieties with $\mathbb{P}^n[k]$-units presented in this article are constructed out of examples of hyperkähler manifolds with special autoequivalences, usually with the property that the quotients are strict Enriques varieties. Then the varieties with $\mathbb{P}^n[k]$-units are constructed as intermediate quotients between the product of the hyperkähler manifolds and the product of the quotients.

It would be very interesting to find ways to construct varieties $X$ with $\mathbb{P}^n[k]$-units directly. In the case $k = 4$, by Proposition 4.5.3, the universal cover of such an $X$ decomposes into two hyperkähler manifolds. Hence, one could hope to find in this way new examples of Enriques or even hyperkähler varieties.

For example, one could try to construct varieties with $\mathbb{P}^n[k]$ units as moduli spaces of sheaves (or objects) on varieties with trivial canonical bundle (or Calabi–Yau categories) of dimension $k$. Indeed, all of the examples that we found in this paper can be realised as moduli spaces.

For example, let $A$, $B$ be abelian surfaces together with automorphisms $a \in \text{Aut}(A)$ and $b \in \text{Aut}(B)$. We set $Y := K_2A$, $Z := K_2B$, $f := K_2a$, $g := K_2b$ and assume that $Y/\langle f \rangle$ and $Z/\langle g \rangle$ are strict Enriques varieties of index 3. This implies that $X := (Y \times Z)/\langle f \times g \rangle$ has a $\mathbb{P}^2[4]$-unit; see Remark 4.4.6. As $Y = K_2A$ and $Z = K_2B$ are moduli spaces of sheaves on $A$ and $B$, respectively, the product $Y \times Z$ is a moduli space of sheaves on $A \times B$. We denote the universal family by $\mathcal{F} \in \text{Coh}(A \times B \times Y \times Z)$. This descends to a sheaf $\mathcal{F} \in \text{Coh}((A \times B)/\langle a \times b \rangle \times X)$ which is flat over $X$ with pairwise non-isomorphic fibres. One can deduce this from the fact that $\mathcal{F}$ is $\langle a \times b \times f \times g \rangle$-linearisable; compare [1, Sect. 3]. Hence, we can consider $X$ as a moduli space of sheaves on $(A \times B)/\langle a \times b \rangle$ with universal family $\mathcal{F}$.

\textbf{References}


Chapter 5

On the derived category of the Hilbert scheme of points on an Enriques surface


Abstract

We use semi-orthogonal decompositions to construct autoequivalences of Hilbert schemes of points on Enriques surfaces and of Calabi–Yau varieties which cover them. While doing this, we show that the derived category of a surface whose irregularity and geometric genus vanish embeds into the derived category of its Hilbert scheme of points.

5.1 Introduction

The bounded derived category of coherent sheaves on a smooth projective complex variety $Z$, denoted by $D^b(Z)$, is now widely recognized as an important invariant which can be used to study the geometry of $Z$. It is therefore quite natural to consider the group of autoequivalences $\text{Aut}(D^b(Z))$. This group always contains the subgroup $\text{Aut}^\text{st}(D^b(Z))$ of so-called standard autoequivalences, namely those generated by automorphisms of $Z$, the shift functor and tensor products with line bundles. Note that all these equivalences send coherent sheaves to (shifts of) coherent sheaves. A classical result by Bondal and Orlov, see [BO01], states that $\text{Aut}^\text{st}(D^b(Z)) \cong \text{Aut}(D^b(Z))$ if $\omega_Z$ is either ample or anti-ample. Therefore, we expect the most interesting behaviour if the canonical bundle is trivial. For instance, if $\omega_Z \cong \mathcal{O}_Z$ and $H^i(Z, \omega_Z) = 0$ for all $0 < i < \dim(Z)$, then $\text{Aut}(D^b(Z))$ contains so-called spherical twists, which are more interesting than the ones mentioned above since, in general, they do not preserve the abelian category of coherent sheaves; see [ST01].

It is usually quite difficult to construct new autoequivalences of a given variety $Z$. However, if $\mathcal{A}$ is some triangulated category and an exact functor $\Phi: \mathcal{A} \to D^b(Z)$ is a so-called spherical or $\mathbb{P}^n$-functor, see Subsection 5.2.5, we do get an autoequivalence of $D^b(Z)$. For example, let $\tilde{X}$ be a K3 surface and $\tilde{X}^{[n]}$ its Hilbert scheme of $n$ points. It was shown in [Add16] that the Fourier–Mukai functor $D^b(\tilde{X}) \to D^b(\tilde{X}^{[n]})$ with the ideal sheaf of the universal family as kernel is a $\mathbb{P}^{n-1}$-functor whose associated autoequivalence is new. Interestingly, for an
abelian surface $A$ the corresponding functor is not a $\mathbb{P}^{n-1}$-functor, but pulling everything to the generalised Kummer variety does give one; see [Mea15]. Another example was given in [6] where it was shown that, given any surface $S$, there is a $\mathbb{P}^{n-1}$-functor $\mathcal{D}^b(S) \to \mathcal{D}^b(S^{[n]})$ which is defined using equivariant methods.

In this paper we will construct new examples of spherical functors using Enriques surfaces and Hilbert schemes of points on them. Let $X$ be an Enriques surface and $X^{[n]}$ the Hilbert scheme of $n$ points on $X$. The canonical cover of $X^{[n]}$ is a Calabi–Yau variety (see [Nie09] or [OS11]) and will be denoted by $\text{CY}_n$. Write $\pi: \text{CY}_n \to X^{[n]}$ for the quotient map. Consider the Fourier–Mukai functor $F: \mathcal{D}^b(X) \to \mathcal{D}^b(X^{[n]})$ induced by the ideal sheaf of the universal family.

**Theorem 5.1.1.** The functor

$$\tilde{F} = \pi^* F: \mathcal{D}^b(X) \to \mathcal{D}^b(\text{CY}_n)$$

is split spherical for all $n \geq 2$ and the associated twist $\tilde{T}$ is equivariant, so descends to an autoequivalence of $X^{[n]}$. The autoequivalence $\tilde{T}$ of $\mathcal{D}^b(\text{CY}_n)$ is not standard and not a twist around a spherical object.

Under some conditions we can also compare our twist $\tilde{T}$ to the autoequivalences constructed in [PS14]; see Proposition 5.3.18.

Once we establish Theorem 5.1.2 below, the first part of the theorem is an incarnation of the following general principle, see Theorem 5.3.4 and Remark 5.3.11:

If $Y$ is a smooth projective variety whose canonical bundle is of order 2, $\tilde{Y}$ its canonical cover with quotient map $\pi: \tilde{Y} \to Y$, and $\mathcal{A}$ an admissible subcategory of $\mathcal{D}^b(Y)$ with embedding functor $i: \mathcal{A} \to \mathcal{D}^b(Y)$, then $\pi^* i$ is a split spherical functor and the associated twist is equivariant.

The following result might be of independent interest.

**Theorem 5.1.2.** If $S$ is any surface with $p_g = q = 0$, then the FM-transform $F: \mathcal{D}^b(S) \to \mathcal{D}^b(S^{[n]})$ whose kernel is the ideal sheaf of the universal family, is fully faithful, hence $\mathcal{D}^b(S)$ is an admissible subcategory of $\mathcal{D}^b(S^{[n]})$.

Since there are many semi-orthogonal decompositions of $\mathcal{D}^b(X^{[n]})$, we have many, potentially non-standard, twists associated with them. Note that in general it seems difficult to describe the complement of the image of $F$ explicitly but see Remark 5.3.14 and Subsection 5.5.3 for statements related to this question.

The paper is organised as follows. In Section 2 we present some background information, before proving our main results in Section 3. In Section 4 we give a general construction of exceptional sequences on the Hilbert scheme $S^{[n]}$ out of exceptional sequences on a surface $S$. This construction has been independently considered by Evgeny Shinder. In particular, we have the following result which is probably well-known to experts.

**Proposition 5.1.3.** If a surface $S$ has a full exceptional collection, then so too does $S^{[n]}$.

In the last section we describe what we call truncated ideal functors which provide us with a further example of a fully faithful functor $\mathcal{D}^b(X) \to \mathcal{D}^b(X^{[n]})$ for $X$ an Enriques surface, and, in some cases, $\mathbb{P}^n$-functors on smooth Deligne–Mumford stacks. The last section also gives some background on the proof of Proposition 5.3.1, the main ingredient in the proof of Theorem 5.1.2.
Conventions. We will work over the complex numbers and all functors are assumed to be derived. Furthermore, all varieties are assumed to be smooth and projective unless stated otherwise. We will write $H^i(E)$ for the $i$-th cohomology object of a complex $E \in D^b(Z)$ and $H^*(E)$ for the complex $\oplus_i H^i(Z, E)[-i]$. If $F$ is a functor, its right adjoint will be denoted by $F^R$ and its left adjoint by $F^L$.

Acknowledgements. We thank Ciarán Meachan and David Ploog for comments. We are very grateful to the referee for many helpful comments which greatly improved the exposition. A.K. was supported by the SFB/TR 45 of the DFG (German Research Foundation). P.S. was partially financially supported by the RTG 1670 of the DFG.

5.2 Preliminaries

5.2.1 Hilbert schemes of surfaces with $p_g = q = 0$

Let $S$ be a surface with $p_g = q = 0$ and consider $S^{[n]}$, the Hilbert scheme of $n$ points on $S$. Then we have $H^k(S^{[n]}, \mathcal{O}_{S^{[n]}}) = 0$ for all $k > 0$, compare [OS11]. Indeed, by the Künneth formula $H^k(S^n, \mathcal{O}_{S^n}) = H^2(S, \mathcal{O}_S)^{\otimes n}$ is concentrated in degree zero. As a consequence, the structure sheaf of the $n$-th symmetric product has no higher cohomology, and the same then also holds for $S^{[n]}$, because the symmetric product has rational singularities.

For example, we can consider an Enriques surface, which is a smooth projective surface $X$ with $p_g = q = 0$ such that the canonical bundle $\omega_X$ is of order 2.

5.2.2 Canonical covers

Let $Y$ be a variety with torsion canonical bundle of (minimal) order $k$. The canonical cover $\tilde{Y}$ of $Y$ is the unique (up to isomorphism) variety with trivial canonical bundle and an étale morphism $\pi: \tilde{Y} \to Y$ of degree $k$ such that $\pi_*\mathcal{O}_{\tilde{Y}} = \bigoplus_{i=0}^{k-1} \omega_Y^i$. In this case, there is a free action of the cyclic group $\mathbb{Z}/k\mathbb{Z}$ on $\tilde{Y}$ such that $\pi$ is the quotient morphism.

As an example, the canonical cover of an Enriques surface $X$ is a K3 surface $\tilde{X}$ and $X$ is the quotient of $\tilde{X}$ by the action of a fixed point free involution.

Furthermore, the canonical bundle of $X^{[n]}$ has order 2 and the associated canonical cover, denoted here by $CY_n$, is a Calabi–Yau variety; see [Nie09, Prop. 1.6] or [OS11, Thm. 3.1].

5.2.3 Fourier–Mukai transforms and kernels

Recall that given an object $E$ in $D^b(Z \times Z')$, where $Z$ and $Z'$ are smooth and projective, we get an exact functor $D^b(Z) \to D^b(Z')$, $\alpha \mapsto p_{Z'}^*(E \otimes p_Z^*\alpha)$. Such a functor, denoted by $\text{FM}_E$, is called a Fourier–Mukai transform (or FM-transform) and $E$ is its kernel. See [Huy06] for a thorough introduction to FM-transforms. For example, $\text{FM}_{\Delta, \mathcal{L}}(\alpha) = \alpha \otimes \mathcal{L}$, where $\Delta: Z \to Z \times Z$ for the diagonal map and $\mathcal{L} \in \text{Pic}(Z)$. In particular, $\text{FM}_{\mathcal{O}_Z}$ is the identity functor.

Convention. We will write $M_{\mathcal{L}}$ for the functor $\text{FM}_{\Delta, \mathcal{L}}$.

Let $S$ be any smooth projective surface, $Z_n \subset S \times S^{[n]}$ be the universal family and consider its structure sequence

$$0 \longrightarrow \mathcal{I}_{Z_n} \longrightarrow \mathcal{O}_{S \times S^{[n]}} \longrightarrow \mathcal{O}_{Z_n} \longrightarrow 0.$$
We can use the objects from the above sequence as kernels to get a triangle $F \to F' \to F''$ of functors $D^b(S) \to D^b(S^{[n]})$. Since all these functors are FM-transforms, they have left and right adjoints; see [Huy06, Prop. 5.9].

5.2.4 Equivalences of canonical covers

The relation between autoequivalences of a variety $Y$ with torsion canonical bundle and those of the canonical cover $\tilde{Y}$ was studied in [BM98]. We recall some facts in the special case where the order of $\omega_Y$ is 2. We will write $\tau$ for the fixed-point free involution of $Y$ such that $\tilde{Y}/(\tau) \cong Y$ and $\pi$ for the quotient map.

An autoequivalence $\tilde{\varphi}$ of $D^b(\tilde{Y})$ is \textit{equivariant} if $\tau_* \tilde{\varphi} \cong \tilde{\varphi} \tau_*$. By [BM98, Sect. 4], an equivariant functor $\tilde{\varphi}$ descends to a functor $\varphi \in \text{Aut}(D^b(Y))$ with functor isomorphisms $\pi_* \tilde{\varphi} \cong \varphi \pi_*$ and $\pi^* \varphi \cong \tilde{\varphi} \pi^*$; moreover, the two descents $\varphi$, $\varphi'$ of $\tilde{\varphi}$ differ by tensoring with $\omega_Y$.

In the other direction, it is also shown in [BM98, Sect. 4] that every autoequivalence of $\text{Aut}(D^b(Y))$ has an equivariant lift. Two lifts differ by the action of $\tau$ in $\text{Aut}(D^b(\tilde{Y}))$.

5.2.5 Spherical functors

Now consider two triangulated categories $A$ and $B$ and any exact functor $F: A \to B$ with left and right adjoints $F^L, F^R: B \to A$. Define the \textit{twist} $T = T_F$ to be the cone on the counit $\epsilon: FF^R \to \text{id}_B$ of the adjunction and the \textit{cotwist} $C$ to be the cone on the unit $\eta: \text{id}_A \to F^RF$.

Remark 5.2.1. Of course, one needs to make sure that the above cones actually exist. If one works with Fourier–Mukai-transforms, this is not a problem, because the maps between the functors come from the underlying kernels and everything works out, even for (reasonable) schemes which are not necessarily smooth and projective; see [AL12]. More generally, everything works out if one uses an appropriate notion of a spherical DG-functor; see [AL13].

So, as we will see, in the cases of interest to us, we have the triangles $FF^R \to \text{id}_B \to T$ and $\text{id}_A \to F^RF \to C$. Following [AL13], we call $F$ \textit{spherical} if $C$ is an equivalence and $F^R \cong CF^L$. If $A$ and $B$ admit Serre functors $S_A$ and $S_B$, the last condition is equivalent to $S_BFC \cong FS_A$. If $F$ is a spherical functor, then $T$ is an equivalence. If the triangle $\text{id}_A \to F^RF \to C$ splits, we call $F$ \textit{split spherical}.

For an example of a (split) spherical functor consider a $d$-dimensional variety $Z$ and a \textit{spherical object} $E \in D^b(Z)$, that is, $E \otimes \omega_Z \cong E$ and $\text{Hom}^*(E, E) \cong \mathbb{C} \oplus \mathbb{C}[-d]$. The functor

$$F = - \otimes E: D^b(\text{Spec}(\mathbb{C})) \to D^b(Z)$$

is then spherical and the associated autoequivalence of $D^b(Z)$ is the spherical twist from the introduction, denoted by $\text{ST}_E$ and called \textit{Seidel–Thomas twist} in the following.

5.2.6 $\mathbb{P}^n$-functors

Following [Add16, Def. 3.1], a \textit{$\mathbb{P}^n$-functor} is a functor $F: A \to B$ of triangulated categories such that

1. There is an autoequivalence $H_F = H$ of $A$ such that

$$F^RF \cong \text{id} \oplus H \oplus H^2 \oplus \cdots \oplus H^n.$$
2. The map
\[ HF^R F \leftrightarrow F^R F F^R F \xrightarrow{\epsilon} F^R F, \]
with \( \epsilon \) being the counit of the adjunction is, when written in the components
\[ H \oplus H^2 \oplus \cdots \oplus H^n \oplus H^{n+1} \rightarrow \text{id} \oplus H \oplus H^2 \oplus \cdots \oplus H^n, \]
of the form
\[
\begin{pmatrix}
* & * & \cdots & * & * \\
1 & * & \cdots & * & * \\
0 & 1 & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & *
\end{pmatrix}
\quad (5.1)
\]

3. \( F^R \simeq H^n F^L \). If \( \mathcal{A} \) and \( \mathcal{B} \) have Serre functors, this condition is equivalent to \( S_B F H^n \simeq F S_A \).

If \( F \) is a \( \mathbb{P}^n \)-functor, then there is also an associated autoequivalence of \( \mathcal{B} \), denoted by \( P_F = P \). A \( \mathbb{P}^1 \)-functor is precisely a split spherical functor and for the associated equivalences we have \( T^2 \cong P \). If \( X \) is a K3 surface, the functor \( F = FM_{\mathbb{I}^n} \) defined in Subsection 5.2.3 is a \( \mathbb{P}^{n-1} \)-functor; see [Add16].

5.2.7 Semi-orthogonal decompositions

References for the following facts are, for example, [Bon89] and [BO95].

Let \( \mathcal{T} \) be a triangulated category. A semi-orthogonal decomposition of \( \mathcal{T} \) is a sequence of strictly full triangulated subcategories \( \mathcal{A}_1, \ldots, \mathcal{A}_m \) such that (a) if \( A_i \in \mathcal{A}_i \) and \( A_j \in \mathcal{A}_j \), then \( \text{Hom}(A_i, A_j[l]) = 0 \) for \( i > j \) and all \( l \), and (b) the \( A_i \) generate \( \mathcal{T} \), that is, the smallest triangulated subcategory of \( \mathcal{T} \) containing all the \( A_i \) is already \( \mathcal{T} \). We write \( \mathcal{T} = \langle A_1, \ldots, A_m \rangle \).

If \( m = 2 \), these conditions boil down to the existence of a functorial exact triangle \( A_2 \rightarrow \tau \rightarrow A_1 \) for any object \( T \in \mathcal{T} \).

A subcategory \( \mathcal{A} \) of \( \mathcal{T} \) is right admissible if the embedding functor \( i \) has a right adjoint \( i^R \), left admissible if \( i \) has a left adjoint \( i^L \), and admissible if it is left and right admissible. Note that if \( \mathcal{T} \) admits a Serre functor, then the existence of one of the adjoints implies the existence of the other.

Given any subcategory \( \mathcal{A} \), the category \( \mathcal{A}^\perp \) consists of objects \( b \) such that \( \text{Hom}(a, b[k]) = 0 \) for all \( a \in \mathcal{A} \) and all \( k \in \mathbb{Z} \). If \( \mathcal{A} \) is right admissible, then \( \mathcal{T} = \langle \mathcal{A}^\perp, \mathcal{A} \rangle \) is a semi-orthogonal decomposition. Similarly, \( \mathcal{T} = \langle \mathcal{A}^\perp, \mathcal{A}^\perp \rangle \) is a semi-orthogonal decomposition if \( \mathcal{A} \) is left admissible, where \( ^\perp \mathcal{A} \) is defined in the obvious way.

Examples typically arise from so-called exceptional objects. Recall that an object \( E \in D^b(Z) \) (or any \( \mathbb{C} \)-linear triangulated category) is called exceptional if \( \text{Hom}(E, E) = \mathbb{C} \) and \( \text{Hom}(E, E[k]) = 0 \) for all \( k \neq 0 \). The smallest triangulated subcategory containing \( E \) is then equivalent to \( D^b(\text{Spec}(\mathbb{C})) \) and this category, by abuse of notation again denoted by \( E \), is admissible, leading to a semi-orthogonal decomposition \( D^b(Z) = \langle E^\perp, E \rangle \). We call a sequence of exceptional objects \( E_1, \ldots, E_n \) an exceptional collection if \( D^b(Z) = \langle (E_1, \ldots, E_n)^\perp, E_1, \ldots, E_n \rangle \), where \( (E_1, \ldots, E_n)^\perp \) is the category of objects \( F \) which satisfy \( \text{Hom}(E_i, F[k]) = 0 \) for all \( i, k \).

The collection is called full if \( (E_1, \ldots, E_n)^\perp = 0 \).
Note that any fully faithful FM-transform $i: \mathcal{A} = \mathbb{D}^b(Z') \to \mathbb{D}^b(Z)$ gives a semi-orthogonal decomposition $\mathbb{D}^b(Z) = \langle (i(A))^-, i(A) \rangle$.

We will need the following well-known and easy fact.

**Lemma 5.2.2.** If $\mathcal{T}$ has a Serre functor $S_{\mathcal{T}}$ and $\mathcal{A}$ is an admissible subcategory, then $\mathcal{A}$ has a Serre functor $S_{\mathcal{A}} \cong i^R S_{\mathcal{T}} i$.

**Proof.** Given $a, a' \in \mathcal{A}$, we compute

$$\text{Hom}_{\mathcal{A}}(a, a') \cong \text{Hom}_{\mathcal{T}}(i(a), i(a')) \cong \text{Hom}_{\mathcal{T}}(i(a'), S_{\mathcal{T}} i(a))^\vee \cong \text{Hom}_{\mathcal{A}}(a', i^R S_{\mathcal{T}} i(a))^\vee.$$ 

\[ \square \]

**Remark 5.2.3.** Assume $\mathcal{A}$ is an admissible subcategory of $\mathbb{D}^b(Z)$. The embedding functor $i$ lifts to DG-enhancements (see, for example, [KL15] for this notion). It can be checked that the adjoints $i^R$ and $i^L$ also lift to so-called DG quasi-functors, which follows, for example, from [KL15, Lem. 4.4 and Prop. 4.10]. So the composition of $i$ with any FM-transform will lift to the DG-level, hence we are in a position to use the results of [AL13] and all required cones of natural transformations will exist.

### 5.2.8 Group actions and derived categories

Let $G$ be a finite group acting on a smooth projective variety $Z$. The **equivariant derived category**, denoted by $\mathbb{D}^b_G(Z)$, is defined as $\mathbb{D}^b(\text{Coh}^G(Z))$, see, for example, [Plo07] for details.

Recall that for every subgroup $H \subset G$ the restriction functor $\text{Res}: \mathbb{D}^b_G(Z) \to \mathbb{D}^b_H(Z)$ has the inflation functor $\text{Inf}: \mathbb{D}^b_H(Z) \to \mathbb{D}^b_G(Z)$ as a left and right adjoint (see e.g. [Plo07, Sect. 1.4]). It is given for $A \in \mathbb{D}^b(Z)$ by

$$\text{Inf}(A) = \bigoplus_{[g] \in H \backslash G} g^* A$$

with the linearisation given by permutation of the summands.

If $G$ acts trivially on $Z$, there is also the functor $\text{triv}: \mathbb{D}^b(Z) \to \mathbb{D}^b_G(Z)$ which equips an object with the trivial $G$-linearisation. Its left and right adjoint is the functor $(-)^G: \mathbb{D}^b_G(Z) \to \mathbb{D}^b(Z)$ of invariants.

**Convention.** When working with Fourier–Mukai transforms, we will frequently identify the functor with its kernel.

### 5.3 Proofs of the main results

#### 5.3.1 Surfaces with $p_g = q = 0$

Recall the FM-transforms from Subsection 5.2.3. To compute $F^R F$ in the examples known so far, one usually works out the various compositions such as, for example, $F^{nR} F'$. In our case, $F^R F$ has a rather simple shape.

**Proposition 5.3.1.** If $S$ is a surface with $p_g = q = 0$, then the composition $F^R F$ is isomorphic to the identity.
Proof. Firstly, we note that, using results in Section 6 of [Mea15] and the isomorphism $H^*(S^{[n]}, O_{S^{[n]}}) \cong \mathbb{C}$, the following holds:

$$
F^\mu_R F' \cong O_{S \times S},
$$

$$
F^\mu_R F' \cong (O_S \boxtimes \omega_S)[2],
$$

$$
F^\mu_R F'' \cong O_\Delta \oplus O_{S \times S},
$$

$$
F^\mu_R F'' \cong (O_S \boxtimes \omega_S)[2].
$$

Next, $F^\mu_R F \cong 0$ and $F^\mu_R F \cong O_\Delta[-1]$. Indeed, the map $F' \to F''$ induces an isomorphism $F^\mu_R F' \to F^\mu_R F''$ and the component $O_{S \times S} \to O_{S \times S}$ of the induced map $F^\mu_R F' \to F^\mu_R F''$ is an isomorphism too; see [Mea15, Sect. 6] or Section 5.5. Hence, the first assertion follows from the triangle $F^\mu_R F \to F^\mu_R F' \to F^\mu_R F''$.

We then consider the triangle $F^\mu_R F \to F^\mu_R F' \to F^\mu_R F''$ and check that the cokernel of the map $F^\mu_R F' \to F^\mu_R F''$ is isomorphic to $O_\Delta$. Indeed, if

$$
\varphi = (\varphi_1, \varphi_2): A \to A \oplus B
$$

is a map in an abelian category such that the first component is an isomorphism (so $\varphi$ is an injection), the cokernel has to be isomorphic to $B$. To see this, just note that the map $A \oplus B \to B$ defined by $(-\varphi_2\varphi^{-1}, id_B)$ satisfies the universal property of the cokernel.

To finally prove the claim, use the triangle $F^\mu_R F \to F^\mu_R F' \to F^\mu_R F''$ and what was proved above.

Proof of Theorem 5.1.2. Since $F^R F \cong id$, $F$ is fully faithful. On the other hand, $F$ has adjoints, so $F(D^b(S))$ is an admissible subcategory of $D^b(S^{[n]})$. \hfill \qed

Remark 5.3.2. The above shows that for any surface with $p_g = q = 0$, the functor $F$ is quite far from being a spherical or a $\mathbb{P}^n$-functor.

Remark 5.3.3. There exist surfaces $S$ of general type with $p_g = q = 0$ such that $D^b(S)$ contains an admissible subcategory whose Hochschild homology is trivial and whose Grothendieck group is finite or torsion, see, for example, [Böh+15]. Therefore, by Theorem 5.1.2, $D^b(S^{[n]})$ also contains such a (quasi-)phantom category.

5.3.2 Application to Enriques surfaces

Theorem 5.3.4. Let $X$ be an Enriques surface, $CY_n$ the canonical cover of $X^{[n]}$, $\pi: CY_n \to X^{[n]}$ the covering map and $\tau: CY_n \to CY_n$ the deck transformation. If $i: A \to D^b(X^{[n]})$ is the embedding of an admissible subcategory $A$ of $D^b(X^{[n]})$, then $\pi^* i$ is a split spherical functor whose induced twist $T_{\pi^* i}$ is equivariant and thus descends to an autoequivalence of $D^b(X^{[n]})$ for all $n \geq 2$.

Proof. First, the functor $\pi^*: D^b(X^{[n]}) \to D^b(CY_n)$ is split spherical with cotwist $C = (-) \otimes \omega_{X^{[n]}}$ and twist $T_{\pi^*} = \tau^*[1]$. Indeed, this follows from the identities $\pi_\ast \pi^* \cong (-) \otimes (O_{X^{[n]}} \oplus \omega_{X^{[n]}})$, $\pi^* \omega_{X^{[n]}} \cong \omega_{CY_n}$, and $\pi^* \pi_\ast \cong id \oplus \tau_\ast$.

Next, $\pi^* i$ is also split spherical as follows from the first step together with Lemma 5.2.2; compare [Add16, Prop. 1.1].

Finally, to see that $T_{\pi^*}$ is equivariant, we note that $\pi^* ii_R \pi_\ast \tau_\ast \cong \pi^* ii_R \pi_\ast \cong \tau_\ast \pi^* ii_R \pi_\ast$, so $\tau_\ast T_{\pi^*} \cong T_{\pi^*} \tau_\ast$, since both are a cone of $\pi^* ii_R \pi_\ast \to \tau_\ast$. \hfill \qed
Example 5.3.5. If $\mathcal{A} \cong \text{D}^b(\text{Spec}(\mathbb{C}))$ is the category generated by an exceptional object $E$, then the twist $\tilde{T}_A$ associated to $\pi^*i$ is the Seidel–Thomas twist $ST_A$, where $A \cong \pi^*i(E)$ is a spherical object by [ST01, Prop. 3.13].

Example 5.3.6. Setting $i = F: \text{D}^b(X) \to \text{D}^b(X^{[n]})$ as the FM transform along the universal ideal sheaf gives the first part of Theorem 5.1.1.

Remark 5.3.7. The functor $\tilde{F}: \text{D}^b(\tilde{X}) \to \text{D}^b(X^{[n]})$ defined as $F\pi X_*$ fails to be spherical in an interesting way. Note that $\pi_X^* = \pi_\tilde{X}^*$ in this case, so $\tilde{R}F \cong \pi_X^*\pi X_* \cong \text{id} \oplus \pi_X^*$ and $C \cong \pi_X^*$ is an autoequivalence of $\text{D}^b(\tilde{X})$. But the condition $\tilde{F}S_\tilde{X} \cong S_{X^{[n]}}\tilde{F}C$ is not satisfied. Indeed, $\tilde{F}S_\tilde{X}(\alpha) \cong \tilde{F}(\alpha)[2]$ whereas $S_{X^{[n]}}\tilde{F}C(\alpha) \cong \tilde{F}(\alpha) \otimes \omega_{X^{[n]}}[2n]$, so these objects are not isomorphic for $\alpha \in \text{D}^b(\tilde{X})$ since their non-vanishing cohomologies lie in different degrees.

One could call $F$ a spherelike functor in analogy with the spherelike objects of [HKP16]. Furthermore, it is easy to check that the functor $F: \text{D}^b(\tilde{X}) \to \text{D}^b(CY_n)$ defined as $\pi^*F\pi X_*$ is not spherical as well.

Remark 5.3.8. Let $B = \perp A$ so that $\text{D}^b(Y) = \langle A, B \rangle$ is a semi-orthogonal decomposition. Then, by [AA13, Thm. 11] we have $\tilde{T}_A T_B \cong \tau^*[1]$.

Remark 5.3.9. One of the two descents of $\tilde{T}_A$ is

$$T_A := \text{cone}(ii^R \oplus M_{\omega_X^{[n]}} ii^R M_{\omega_X^{[n]}} \xrightarrow{(\epsilon \epsilon)} \text{id})$$

(5.3)

where $\epsilon$ is the counit of the adjunction. To see this, first note that the twist $\tilde{T}_A$ along the spherical functor $\pi^*i$ is given by

$$\tilde{T}_A := \text{cone}(\pi^*ii^R \pi_* \xrightarrow{\epsilon} \text{id}).$$

Since $\pi_*\pi^* \cong \text{id} \oplus M_{\omega_X^{[n]}}$ and $\pi^*M_{\omega_X^{[n]}} \cong \pi^*$, we have

$$(\pi^*ii^R \pi_*)^* \cong \pi^*ii^R \oplus \pi^*ii^R M_{\omega_X^{[n]}} \cong \pi^*(ii^R \oplus M_{\omega_X^{[n]}} ii^R M_{\omega_X^{[n]}})$$

from which $\tilde{T}_A \pi^* \cong \pi^* T_A$ follows. Using $M_{\omega_X^{[n]}} \pi_* \cong \pi_*$ one can similarly show that $\pi_* \tilde{T}_A \cong T_A \pi_*$.

Remark 5.3.10. As in the case of spherical or $\mathbb{P}^n$-twists (see [6, Lem. 2.3]), we have for any $\Psi \in \text{Aut}(\text{D}^b(X^{[n]}))$ the relation $\Psi T_A \cong T_{\Psi(A)} \Psi$. For this one uses the cone description of $T_A$ given by Equation (5.3) and the fact that the embedding functor of $\Psi(A)$ is $\Psi i$.

Remark 5.3.11. The reader will note that the proof of Theorem 5.3.4 shows more generally that the functor $\pi^*i$ is split spherical with equivariant twist for any canonical cover $\pi: \tilde{Y} \to Y$ of degree 2 and any fully faithful admissible embedding $i: \mathcal{A} \to \text{D}^b(Y)$. More generally, $\pi^*$ is a $\mathbb{P}^{n-1}$-functor if $\pi: \tilde{Y} \to Y$ is a canonical cover of order $n \geq 2$. But for $n \geq 3$ the composition $\pi^*i$ is in general not a $\mathbb{P}^{n-1}$-functor for a fully faithful admissible embedding $i: \mathcal{A} \to \text{D}^b(Y)$. The reason is that Lemma 5.2.2 does not generalise to powers of the Serre functor, that is, in general, $S_A^k \neq i^R S_F^k i$ for $k \geq 2$. 
5.3.3 Comparison to known autoequivalences

Let \( X \) be an Enriques surface and \( F : \mathcal{D}^b(X) \to \mathcal{D}^b(X[n]) \) the FM-transform induced by the ideal sheaf of the universal family. We denote the twist along the spherical functor \( \bar{F} = \pi^*F \) by \( \bar{T} \in \text{Aut}(\mathcal{D}^b(CY_n)) \) and its descent as described in Remark 5.3.9 by \( T \in \text{Aut}(\mathcal{D}^b(X[n])) \).

The second descent is given by \( M_{\omega_{X[n]}} T \). Now we want to compare these new autoequivalences to the known ones. Of course, on any variety there are the standard autoequivalences. On \( CY_n \) there are also Seidel-Thomas twists and the equivalences constructed in [PS14], while on \( X[n] \) there is the \( \mathbb{P}^{n-1} \)-functor constructed in [6]. We will need the following statement.

**Lemma 5.3.12.** Let \( S \) be any surface and let \( k(\xi) \in \mathcal{D}^b(S[n]) \) be the skyscraper sheaf of a point \( \xi \in S[n] \). Then \( FFR(k(\xi)) \) has rank \( \chi - 2n \) where \( \chi := \chi(\omega_S) = \chi(\mathcal{O}_S) \).

**Proof.** The kernel of \( FR \) is given by \( I_{\xi} \otimes \mathcal{O}_S[2] \). Since \( I_{\xi} \) is flat over \( X[n] \) we get \( FFR(k(\xi)) = I_{\xi} \otimes \omega_S[2] \) where \( \xi \) on the left-hand side denotes a point of \( S[n] \) and \( \xi \) on the right-hand side denotes the subscheme of \( S \) represented by this point. Note that for any object \( \mathcal{E} \in \mathcal{D}^b(M) \) on a smooth variety \( M \) and \( x \in M \) we have \( \text{rk} \mathcal{E} = \chi(\mathcal{E}, k(x)) \).

Thus,

\[
\text{rk} FFR(\alpha) = \chi(FFR(\alpha), \alpha) = \chi(\alpha, FFR(\alpha)) = \chi(\mathcal{I}_\xi, \mathcal{I}_\xi) = \chi - n - n + 0
\]

where the last equality uses the equation \( \mathcal{I}_\xi = \mathcal{O}_S - \mathcal{O}_\xi \) in the Grothendieck group \( K_0(S) \). \( \square \)

**Proposition 5.3.13.** The autoequivalence \( T \) is not contained in the subgroup of \( \text{Aut}(\mathcal{D}^b(X[n])) \) generated by the standard autoequivalences \( \text{Aut}^\text{st}(\mathcal{D}^b(X[n])) \) and the equivalence \( P \) arising from the \( \mathbb{P}^{n-1} \)-functor constructed in [6]. Similarly, \( \bar{T} \notin (\text{Aut}^\text{st}(\mathcal{D}^b(CY_n)), \bar{P}) \), where \( \bar{P} \) is a lift of \( P \).

**Proof.** The first assertion follows from the second, since standard autoequivalences lift to standard autoequivalences.

The twist \( P \) is rank-preserving since all the objects in the image of the corresponding \( \mathbb{P}^{n-1} \)-functor are supported on a proper subset of \( X[n] \); see [6, Rem. 4.7]. Thus, the lift \( \bar{P} \) is rank preserving too. The same holds for every standard autoequivalence (up to the sign -1 for odd shifts). But by Lemma 5.3.12 we have for \( \xi \in CY_n \) with \( \xi := \pi(\xi) \in S[n] \) the equalities

\[
2 \cdot \text{rk}(\bar{T}(k(\xi))) = \text{rk}(T(k(\xi))) = 4n - 2 \neq 0. \quad \square
\]

Before proceeding further with the comparisons, we need to compute the values of \( \bar{T} \) and \( T \) on certain spanning classes (for details on spanning classes see [Huy06, Sect. 1.3]). Following the proof of Theorem 5.3.4 we see that the cotwist of \( \bar{F} \) is \( C = S_X[-2n] = M_{\omega_X}[2 - 2n] \). By [Add16, Sect. 1.4] \( \bar{T} \) is given on the spanning class \( \text{im}(\bar{F}) \cup \text{im}(\bar{F})^\perp \) by

\[
\bar{T}F \cong \bar{F}C[1] \cong \bar{F}M_{\omega_X}[3 - 2n], \quad \bar{T}(\beta) = \beta \quad \text{for } \beta \in \text{im}(\bar{F})^\perp.
\]

Note that \( \text{im}(\bar{F})^\perp = \ker(\bar{F}R) = \{ \beta \in \mathcal{D}^b(CY_n) \mid \bar{F}R(\beta) = 0 \} \).

Using the cone description (5.3) of \( T \) we also get

\[
TF \cong M_{\omega_{X[n]}} \mathcal{F}_{\omega_X}[3 - 2n], \quad TM_{\omega_{X[n]}} F \cong \mathcal{F}_{\omega_X}[3 - 2n]
\]

by Lemma 5.2.2. Also, \( T \) acts as the identity on \( \langle A, A \otimes \omega_{X[n]} \rangle^\perp = \ker(F^R) \cap \ker(F^L) \) where \( A = \text{im}(F) \). So we have a description of the restriction of \( T \) to the spanning class.
Lemma 5.3.17. Let $\beta \in \langle A, \mathcal{A} \otimes \omega_X[n] \rangle$ and $\alpha \in \mathcal{A}$. By Serre duality, $\beta \in \langle A, \mathcal{A} \otimes \omega_X[n] \rangle$ and $\alpha \in \mathcal{A}$. By Serre duality, $\beta \in \langle A, \mathcal{A} \otimes \omega_X[n] \rangle$ and $\alpha \in \mathcal{A}$. By Serre duality, $\beta \in \langle A, \mathcal{A} \otimes \omega_X[n] \rangle$ and $\alpha \in \mathcal{A}$. By Serre duality, $\beta \in \langle A, \mathcal{A} \otimes \omega_X[n] \rangle$ and $\alpha \in \mathcal{A}$.

Remark 5.3.14. We will see in Section 5.5.3 that there exist non-trivial objects in $\ker(F^R) \cap \ker(F^L)$. By applying $\pi^*$ we also get non-trivial objects in $\ker(F^R) \cap \ker(F^L)$. So there are 0 $\neq \beta \in \mathcal{D}(X[n])$ and 0 $\neq \beta \in \mathcal{D}(CY_n)$ such that $T(\beta) = \beta$ and $T(\beta) = \beta$.

Lemma 5.3.15. Let $E \in \mathcal{D}(CY_n)$ be a spherical object and $ST_E$ the induced Seidel–Thomas twist. Let 0 $\neq \alpha \in \mathcal{D}(CY_n)$ with $ST_E(\alpha) = \alpha[\ell]$. Then $\ell = 0$ or $\ell = 1 - 2n$.

Proof. We have $ST_E(E) = E[1 - 2n]$ and $ST_E(\beta) = \beta$ for all $\beta \in E^\perp$. The assertion follows by [Add16, Prop. 1.2] together with the fact that $E \cup E^\perp$ is a spanning class of $\mathcal{D}(CY_n)$. See also Lemma 5.3.17 below for a similar statement with a similar proof.

Proposition 5.3.16. The twist $T$ does not equal a shift of a Seidel–Thomas twist.

Proof. By (5.4) we have $T(\alpha) = \alpha[3 - 2n]$ for $\alpha \in \mathcal{D}(X)$ with $\omega_X \otimes \alpha \cong \omega_X$. a skyscraper sheaf. By Remark 5.3.14 there is also a 0 $\neq \beta \in \mathcal{D}(CY_n)$ such that $T(\beta) = \beta$. Thus, the assumption that $T[m] = ST_E$ for some $m \in \mathcal{Z}$ contradicts the previous lemma.

Recall that if $Z, Z'$ are smooth projective varieties and $a < b$ two integers, an exact functor $G: \mathcal{D}(Z) \to \mathcal{D}(Z')$ is said to have cohomological amplitude $[a, b]$ if for every complex $E \in \mathcal{D}(Z)$ whose cohomology is concentrated in degrees between $p$ and $q$, the cohomology of $G(E)$ is concentrated in degrees between $p - a$ and $q + b$. We will need the following result.

Lemma 5.3.17. Let 0 $\neq \alpha \in \mathcal{D}(X[n])$ with $T(\alpha) = \alpha[\ell]$ or $T(\alpha) = \alpha \otimes \omega_X[n][\ell]$. Then $\ell = 0$ or $\ell = 3 - 2n$. Similarly, if $\gamma \in \mathcal{D}(CY_n)$ with $\gamma[\ell]$ then $\ell = 0$ or $\ell = 3 - 2n$.

Proof. Let $\alpha \in \mathcal{D}(X[n])$ with $T(\alpha) = \alpha[\ell]$, $\ell \notin \{0, 3 - 2n\}$. To simplify notation write $\Phi = M_{\omega_X[n]}$, $\Psi = M_{\omega_X}$ and $j = 3 - 2n$. By Equation (5.5) we have $TF \cong \Phi F \Psi[j]$. Hence, for any $\beta \in \mathcal{D}(X)$ we have

$$
\text{Hom}(\alpha, F(\beta)[k]) \cong \text{Hom}(T^m(\alpha), T^m F(\beta)[k]) \\
\cong \text{Hom}(\alpha[m\ell], \Phi^m F \Psi^m(\beta)[mj + k]) \\
\cong \text{Hom}(\Psi^{-m} F^L \Phi^{-m}(\alpha), \beta[(j - \ell)m + k])
$$

for every $k \in \mathcal{Z}$. This vanishes for $m \gg 0$ since $\Phi = M_{\omega_X[n]}$ and $\Psi = M_{\omega_X}$ have cohomological amplitude $[0, 0]$ and $F^L$ has finite cohomological amplitude by [Kuz06, Prop. 2.5]. Therefore, $\alpha \in \mathcal{A}$ where $A = \text{im } F$. Similarly, we get $\alpha \in \mathcal{A} \otimes \omega_X[n]$. The object $\alpha$ is also orthogonal to $\mathcal{B} := \langle A, \mathcal{A} \otimes \omega_X[n] \rangle$, on which $T$ acts trivially. Indeed, for $\beta \in \mathcal{B}$ and $k \in \mathcal{Z}$ we have

$$
\text{Hom}(\alpha, \beta[k]) = \text{Hom}(T^m(\alpha), T^m(\beta)[k]) = \text{Hom}(\alpha[m\ell], \beta[k])
$$

which vanishes for $m \gg 0$.

Therefore, $\alpha$ is orthogonal to the spanning class $\mathcal{A} \cup \mathcal{A} \otimes \omega_X[n] \cup \langle A, \mathcal{A} \otimes \omega_X[n] \rangle$, hence is zero. The proof in the case that $T(\alpha) = \alpha \otimes \omega_X[n][\ell]$ is similar and the statement about $T$ and $\gamma$ follows by applying $\pi_*$.

\end{document}
Now we want to compare our autoequivalence to those described in [PS14]. While recalling the construction of the latter, we also introduce some more general facts which will be useful in the next section.

Let $Z$ be a smooth projective variety and $n \geq 2$. We consider the cartesian power $Z^n$ equipped with the natural $\mathfrak{S}_n$-action given by permuting the factors.

For $E \in \mathcal{D}^b(Z)$ an exceptional object, the box product $E^{\otimes n}$ is again exceptional, since by the K"{a}hler--Zumeth formula

$$\text{Ext}^*_E(Z^n)(E^{\otimes n}, E^{\otimes n}) \cong \text{Ext}^*_e(E^{\otimes n}, E^{\otimes n}) \cong S^n \text{Ext}^*_X(E, E) \cong \mathbb{C}[0].$$

More generally, for $\varrho$ an irreducible representation of $\mathfrak{S}_n$, the object $E^{\otimes n} \otimes \varrho$ is exceptional and the objects obtained this way are pairwise orthogonal, i.e. $\text{Ext}^*(E^{\otimes n} \otimes \varrho, E^{\otimes n} \otimes \varrho') = 0$ for $\varrho \neq \varrho'$.

The case of biggest interest is if $Z$ is a surface. Then there is the Bridgeland--King--Reid--Haiman equivalence (see [BKR01] and [Hai01])

$$\Phi : \mathcal{D}^b(Z[n]) \xrightarrow{\sim} \mathcal{D}^b_Z(Z^n).$$

In particular, when $Z = X$ is an Enriques surface, we get for every exceptional object $E \in \mathcal{D}^b(X)$ further induced autoequivalences given by the Seidel--Thomas twists $\mathcal{T}_\varrho := \text{ST}_{\pi(\varrho)}(E^{\otimes n} \otimes \varrho) \in \text{Aut}(\mathcal{D}^b(CY_n))$ and their descents $\mathcal{T}_\varrho \in \text{Aut}(\mathcal{D}^b(X[n]))$. We assume from now on that there exists an object $0 \neq F \in \langle E, E \otimes \omega_X \rangle^\perp$. This is equivalent to $\mathcal{E}$ being non-trivial, where $\mathcal{E} := \pi^*E$ is the corresponding spherical object on the K3 cover $\mathcal{X}$ of $X$ (for example, if $E$ is a line bundle, we can take $\mathcal{E} \otimes \mathcal{T}_x \otimes I_y \in \mathcal{E} \otimes I_y$ for two distinct points $x, y$). In this case $E^{\otimes n}$ is orthogonal to every $E^{\otimes n} \otimes \varrho$ and $\omega_X \otimes E^{\otimes n} \otimes \varrho$. Thus, $\mathcal{T}_\varrho(\alpha) = \alpha$ for $\alpha = \pi^*\Phi^{-1}(E^{\otimes n})$. Since also $\mathcal{T}_\varrho(\beta) = \beta[1 - 2n]$ for $\beta = \pi^*\Phi^{-1}(E^{\otimes n} \otimes \varrho)$ we see that $\mathcal{T}_\varrho$ and thus also its descent $\mathcal{T}_\varrho$ are non-standard.

By construction, the object $\mathcal{E}$ is invariant under $\tau_X$, hence the associated Seidel--Thomas twist descends to an equivalence $\Phi_E$ of $\mathcal{D}^b(X)$. In turn, this gives an equivalence $\Phi^{\otimes n} = \text{FM}_{\mathcal{P}^{\otimes n}} \in \text{Aut}(\mathcal{D}^b(X[n]))$, where $\mathcal{P} \in \mathcal{D}^b(X \times X)$ denotes the Fourier--Mukai kernel of $\Phi_E$; see [Plo07]. It is possible to lift this equivalence to $\Phi^{\otimes n} \in \text{Aut}(CY_n)$; see [PS14] for details.

Recall that the isomorphism classes of irreducible representations of $\mathfrak{S}_n$ are in bijection to the set $P(n)$ of partitions of $n$. For an exceptional object $E \in \mathcal{D}^b(X)$ we set

$$G_E := \langle [1], \Phi^{\otimes n}_E, \mathcal{T}_\varrho : \varrho \in P(n) \rangle \subset \text{Aut}(\mathcal{D}^b(X[n])),$$

$$\hat{G}_E := \langle [1], \Phi^{\otimes n}_E, \mathcal{T}_\varrho : \varrho \in P(n) \rangle \subset \text{Aut}(\mathcal{D}^b(CY_n)).$$

**Proposition 5.3.18.** Let $E \in \mathcal{D}^b(X)$ be exceptional and $0 \neq F \in \langle E, E \otimes \omega_X \rangle^\perp$. We have $G_E \cong \mathbb{Z}^{p(n)+2} \cong \hat{G}_E$, where $p(n) = |P(n)|$. Furthermore, $T \notin G_E$ and $\mathcal{T} \notin \hat{G}_E$.

**Proof.** For $0 \leq k \leq n$ we consider the objects

$$E^k \cdot F^{n-k} := \text{linc}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}(E^{\otimes k} \boxtimes F^{\otimes n-k}) \in \mathcal{D}^b_X(X^n).$$

We have $\Phi(E) = E[-1]$ and $\Phi(F) = F$. It follows that

$$\Phi^{\otimes n}_E : E^k \cdot F^{n-k} \mapsto E^k \cdot F^{n-k}[-k], \quad E^{\otimes n} \otimes \varrho \mapsto E^{\otimes n} \otimes \varrho[-n].$$

(5.6)
By Remark 5.3.10, the latter shows that $\Phi_{E}^{\otimes n}$ commutes with all the $T_{\psi}$. Note that for $k < n$ and $\varrho \neq \varrho'$ we have
\[
\text{Ext}^{*}(E_{\otimes n}^{\otimes} \varrho, E^{\otimes k} \cdot F^{n-k}) = 0 = \text{Ext}^{*}(E_{\otimes n}^{\otimes} \varrho, E_{\otimes n}^{\otimes} \varrho') .
\]
Similarly to (5.5) we get by the cone description (5.3) of $T_{\psi}$
\[
T_{\psi}: E^{\otimes k} \cdot F^{n-k} \mapsto E^{\otimes k} \cdot F^{n-k} \text{ for } k < n ,
\]
\[
T_{\psi}: E_{\otimes n}^{\otimes} \varrho' \mapsto \begin{cases} E_{\otimes n}^{\otimes} \varrho' & \text{if } \varrho \neq \varrho' \\ E_{\otimes n}^{\otimes} \varrho'[-(2n-1)] & \text{if } \varrho = \varrho'
\end{cases}
\]
which, in particular, shows that the $T_{\psi}$ pairwise commute.

Now consider $\Psi = (\Phi_{E}^{\otimes n})^{a} \circ \prod_{c} T_{\psi}^{b_{c}}[c] \in G_{E}$ and assume that $\Psi \cong \text{id}$. We have $\Psi(F_{\otimes n}^{\otimes} \varrho) = F_{\otimes n}^{\otimes}[c]$ and thus $c = 0$. It follows that $\Psi(E^{1} \cdot F^{n-1}) = E^{1} \cdot F^{n-1}[-a]$ and thus $a = 0$. Finally, $\Psi(E_{\otimes n}^{\otimes} \varrho') = \omega_{X^{n}}^{\otimes} E_{\otimes n}^{\otimes} \varrho'[-b_{\varrho}(2n-1)]$ shows that $b_{\varrho} = 0$ for all $\varrho$. The assertion that $T \notin G_{E}$ follows in a similar way using Lemma 5.3.17.

The identities (5.6) and (5.7) lift to identities in $D^{b}(\text{CY}_{n})$. Therefore, one can analogously show that $\bar{G}_{E} \cong \mathbb{Z}^{(n)+2}$ and $\bar{T} \notin G_{E}$.

\begin{remark}
Let $a$ be the sign representation, i.e. the one-dimensional representation on which $\mathfrak{S}_{n}$ acts by multiplication by sgn. By Remark 5.3.10 we have $T_{a} \cong M_{a} \circ T_{\varrho} \circ M_{a}$ where $\mathbb{C}$ denotes the trivial representation and $M_{a} \in \text{Aut}(D^{b}_{\mathfrak{S}_{n}}(X^{n}))$ is the involution $(-) \otimes a$. Note that for higher dimensional irreducible representations $\varrho$ there is no such relation since $M_{\varrho}$ is not an equivalence.
\end{remark}

### 5.4 Exceptional sequences on $X^{[n]}$

Let $Z$ be a smooth projective variety and $n \geq 2$. In this section we will construct exceptional sequences in the equivariant derived category $D^{b}_{\mathfrak{S}_{n}}(Z^{n})$ out of exceptional sequences in $D^{b}(Z)$.

**Proposition 5.4.1** ([Sam07, Cor. 1]). Let $Z$ and $Z'$ be smooth projective varieties with full exceptional sequences $E_{1}, \ldots, E_{k}$ and $F_{1}, \ldots, F_{\ell}$ respectively. Then
\[
E_{1} \boxtimes F_{1}, E_{1} \boxtimes F_{2}, \ldots, E_{1} \boxtimes F_{\ell}, E_{2} \boxtimes F_{1}, \ldots, E_{k} \boxtimes F_{\ell}
\]
is a full exceptional sequence of $D^{b}(Z \times Z')$.

Let now $E_{1}, \ldots, E_{k}$ be an exceptional sequence on $Z$. We consider for every multi-index $\alpha = (\alpha_{1}, \ldots, \alpha_{n}) \in [1, k]^{n} := \{1, \ldots, k\}^{n}$ the object
\[
E(\alpha) := E_{\alpha_{1}} \boxtimes \cdots \boxtimes E_{\alpha_{n}} \in D^{b}(Z^{n}).
\]

**Remark 5.4.2.** Let $\alpha, \beta \in [1, k]^{n}$. By the Künneth formula
\[
\text{Ext}^{*}(E(\alpha), E(\beta)) = \text{Ext}^{*}(E_{\alpha_{1}}, E_{\beta_{1}}) \otimes \cdots \otimes \text{Ext}^{*}(E_{\alpha_{n}}, E_{\beta_{n}}).
\]
Therefore, if the original sequence is full then, by Proposition 5.4.1, these objects form a full exceptional sequence of $D^{b}(Z^{n})$ when considering them with the ordering given by the lexicographical order $\preceq_{\text{lex}}$ on $[1, k]^{n}$. Thus, we have $\text{Ext}^{*}(E(\alpha), E(\beta)) = 0$, whenever $\alpha_{i} > \beta_{i}$ for some $i \in [1, n]$. 

\[142\]
**Theorem 5.4.3 ([Ela09]).** Let $G$ be a finite group acting on a variety $M$. Consider an exceptional sequence of $D^b(M)$ of the form

$$E_1^{(1)}, \ldots, E_{k_1}^{(1)}, E_1^{(2)}, \ldots, E_{k_2}^{(2)}, \ldots, E_1^{(\ell)}, \ldots, E_{k_\ell}^{(\ell)}$$

such that $G$ acts transitively on every block $E_1^{(i)}, \ldots, E_{k_i}^{(i)}$, i.e. for every $i \in [\ell]$ and every pair $a, b \in [1, k_i]$ there is a $g \in G$ such that $g^* E_a^{(i)} \cong E_b^{(i)}$ (and conversely for every $g \in G$ and every $a \in [1, k_i]$ there is an element $b \in [1, k_i]$ such that $g^* E_a^{(i)} \cong E_b^{(i)}$). Let $H_i := \text{Stab}_G(E_1^{(i)})$ and assume that $E_1^{(i)}$ carries an $H_i$-linearisation, i.e. there exists an $E^{(i)} \in D_{H_i}^b(M)$ such that $\text{Res}(E^{(i)}) = E_1^{(i)}$. Then

$$\text{Inf}_{H_1}^G(E^{(1)} \otimes V_1^{(1)}), \ldots, \text{Inf}_{H_1}^G(E^{(1)} \otimes V_{m_1}^{(1)}), \ldots,$$

$$\text{Inf}_{H_\ell}^G(E^{(\ell)} \otimes V_1^{(\ell)}), \ldots, \text{Inf}_{H_\ell}^G(E^{(\ell)} \otimes V_{m_\ell}^{(\ell)})$$

is an exceptional sequence of $D_G^b(M)$ with $V_1^{(i)}, \ldots, V_{m_i}^{(i)}$ being all the irreducible representations of $H_i$. The induced exceptional sequence of $D_G^b(M)$ is full if and only if the original exceptional sequence of $D^b(M)$ is full.

**Proof.** In [Ela09] the Theorem is only stated in the case of full exceptional sequences. But one can easily infer from the proof that non-full exceptional sequences also induce non-full exceptional sequences. \hfill \Box

In order to apply Theorem 5.4.3 we have to reorder the sequence consisting of the $E(\alpha)$ as follows. For a multi-index $\alpha \in [1,k]^n$ we denote the unique non-decreasing representative of its $\mathfrak{S}_n$-orbit by $\text{nd}(\alpha)$. Then we define a total order $\prec$ of $[1,k]^n$ by

$$\alpha \prec \beta : \iff \begin{cases} \text{nd}(\alpha) \leq_{\text{lex}} \text{nd}(\beta) \text{ or } \\ \text{nd}(\alpha) = \text{nd}(\beta) \text{ and } \alpha <_{\text{lex}} \beta \end{cases}$$

Now the group $\mathfrak{S}_n$ acts transitively on the blocks consisting of all $E(\alpha)$ with fixed $\text{nd}(\alpha)$ because of $\sigma^* E(\alpha) \cong E(\sigma^{-1} \cdot \alpha)$. Furthermore, every $E(\alpha)$ has a canonical Stab($\alpha$)-linearisation given by permutation of the factors in the box product. It remains to show that $(E(\alpha))_\alpha$ with the ordering given by $\prec$ is still an exceptional sequence. This follows by Remark 5.4.2 and the last item of the following lemma.

**Lemma 5.4.4.** Let $\alpha, \beta \in [1,k]^n$.

1. Let $\text{nd}(\alpha) = \text{nd}(\beta)$ but $\alpha \neq \beta$. Then there exists an $i \in [n]$ such that $\alpha_i < \beta_i$.

2. Let $\sigma \in \mathfrak{S}_n$. Then there exists an $i \in [1,n]$ such that $\alpha_i < \beta_i$ if and only if there exists an $j \in [1,n]$ such that $(\sigma \cdot \alpha)_j < (\sigma \cdot \beta)_j$.

3. If $\text{nd}(\alpha) <_{\text{lex}} \text{nd}(\beta)$, then there exists an $i \in [1,n]$ with $\alpha_i < \beta_i$.

4. Let $\alpha \prec \beta$. Then there exists an $i \in [1,n]$ such that $\alpha_i < \beta_i$.
Proof. If \( \text{nd}(\alpha) = \text{nd}(\beta) \), we have \( \alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_n \). This shows (1). By setting \( j = \sigma(i) \) we obtain (2). In order to show (3) we may now assume using (2) that \( \alpha = \text{nd}(\alpha) \).

Let \( \sigma \in \mathfrak{S}_n \) be such that \( \beta = \sigma \cdot \text{nd}(\beta) \) and let \( m := \min\{\ell \in [1, n] \mid \text{nd}(\alpha)_{\ell} \neq \text{nd}(\beta)_{\ell}\} \). Then \( \text{nd}(\alpha)_m < \text{nd}(\beta)_m \). If \( \sigma(m) \leq m \), we have \( \alpha_{\sigma(m)} = \text{nd}(\alpha)_{\sigma(m)} \leq \text{nd}(\alpha)_m < \text{nd}(\beta)_m = \beta_{\sigma(m)} \).

If \( \sigma(m) > m \), there exists \( \ell > m \) such that \( \sigma(\ell) \leq m \). This yields

\[
\alpha_{\sigma(\ell)} = \text{nd}(\alpha)_{\sigma(\ell)} \leq \text{nd}(\alpha)_m < \text{nd}(\beta)_m \leq \text{nd}(\beta)_{\ell} = \beta_{\sigma(\ell)}.
\]

Finally, (4) follows from (1) and (3). \( \square \)

We summarize all of the above in the following

**Proposition 5.4.5.** If \( \alpha \in [1, k]^n \) is a non-decreasing multi-index and \( V_i^{(\alpha)} \) is an irreducible representation of \( H_n = \text{Stab}(\alpha) \), then the collection of objects \( \mathcal{E}(\alpha, V_i^{(\alpha)}) := \text{Inf}_{H_n} E(\alpha) \otimes V_i^{(\alpha)} \) forms an exceptional sequence of \( D_{\mathfrak{S}_n}(Z^n) \). The induced sequence is full if and only if the original sequence on \( D^b(Z) \) is full. \( \square \)

**Remark 5.4.6.** An exceptional sequence is called *strong* if all the higher extension groups between its members vanish. Using Equation (5.8) one can show that \( (\mathcal{E}(\alpha, V_i^{(\alpha)}))_{\alpha, i} \) is strong if and only if \( (E_{\ell})_{\ell} \) is strong. Thus, in the case that the full exceptional sequence \( E_1, \ldots , E_k \) of \( D^b(Z) \) is strong, there is an equivalence of triangulated categories \( D_{\mathfrak{S}_n}(Z^n) \cong D^b(\text{Mod} - \text{End}_{\mathfrak{S}_n} (\mathcal{M})) \) where \( \mathcal{M} := \bigoplus_{\alpha, i} \mathcal{E}(\alpha, V_i^{(\alpha)}) \); see [Bon89].

**Remark 5.4.7.** Using [Ela12] one can also construct semi-orthogonal decompositions of \( D_{\mathfrak{S}_n}(Z^n) \) out of general semi-orthogonal decompositions of \( D^b(Z) \) in a similar way. Let us describe the special case that \( D^b(Z) = (\mathcal{A}, E) \) with \( E \in D^b(Z) \) an exceptional object. Then there is an induced semi-orthogonal decomposition of \( D_{\mathfrak{S}_n}(Z^n) \) with components

\[
B(k, g) := \langle \text{Inf}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}} ((E^{\otimes k} \otimes g) \boxtimes B) \mid \text{Res}(B) \cong A_{k+1} \boxtimes \cdots \boxtimes A_n, A_i \in \mathcal{A} \rangle
\]

for \( k = 0, \ldots , n \) and \( g \) an irreducible representation of \( \mathfrak{S}_k \). Note that \( B(n, g) \) is spanned by the exceptional object \( E^{\otimes n} \otimes g \) and \( B(n-1, g) \cong \mathcal{A} \).

**Remark 5.4.8.** If \( B \in D^b(Z) \) is a tilting object, so is \( \oplus_{g \in P(n)} (B^{\otimes n} \otimes g) \in D_{\mathfrak{S}_n}(Z^n) \).

**Remark 5.4.9.** Any Enriques surface has a completely orthogonal exceptional sequence \( (E_i) \) of length 10 consisting of line bundles; see [Zub97]. The induced exceptional sequence in \( D_{\mathfrak{S}_n}(X^n) \cong D^b(X^{[n]}) \) is again completely orthogonal and the same holds for the corresponding sequence of spherical objects on \( CY_n \). By [6, Cor. 2.4] it follows that the associated spherical twists give an embedding \( \mathbb{Z}^{\ell(10,n)} \hookrightarrow \text{Aut}(D^b(CY_n)) \) where \( \ell(10,n) \) denotes the length of the induced sequence. By arguments similar to those in the proof of Proposition 5.3.18 we also get an embedding \( \mathbb{Z}^{\ell(10,n)} \hookrightarrow \text{Aut}(D^b(X^{[n]})) \).

### 5.5 The truncated universal ideal functor

The arguments in this section follow those of [Mea15, Sect. 6]. Recall that for any surface \( Z \) and any \( n \geq 2 \) there is the Bridgeland–King–Reid–Haiman equivalence

\[
\Phi : D^b(Z^{[n]}) \xrightarrow{\sim} D_{\mathfrak{S}_n}^b(Z^n).
\]

144
Note that $Z^{[n]}$ is smooth under our assumptions, while this is not true anymore when $\dim Z \geq 3$ and $n \geq 3$. The key new observation is that the functor $\hat{F} := \Phi F : D^b(Z) \to D^b_{\mathcal{S}_n}(Z^n)$ for $Z = S$ a surface can be truncated to a functor $G$ in such a way that $G^R G \cong \hat{F}^R \hat{F}$ and that this functor generalises in a nicer way to varieties of arbitrary dimension than $\hat{F}$ does.

If $Z = S$ is a surface, then $\hat{F}'' = \Phi FM_{O_{Z^n}} = FM_{\mathcal{C}^*}$, where $\mathcal{C}^*$ is the complex concentrated in degrees $0, \ldots, n - 1$ given by

$$0 \to \bigoplus_{i=0}^{n} O_{D_i} \to \bigoplus_{|I|=2} O_{D_I} \otimes a_I \to \bigoplus_{|I|=3} O_{D_I} \otimes a_I \cdots \to O_{D_{[n]}} \otimes a_{[n]} \to 0;$$

see [Sca09]. For $I \subset [1, n] := \{1, \ldots, n\}$, the reduced subvariety $D_I \subset S \times S^n$ is given by $D_I = \{(y, x_1, \ldots, x_n) \mid y = x_i \forall i \in I\}$ and $a_I$ is the sign representation of $S_I$. Furthermore, $\hat{F}'' = \Phi FM_{O_{S \times S^n}} \cong FM_{O_{S \times S^n}}$ and the induced map $\hat{F}'' \to \hat{F}''$ is given by the morphism of kernels $O_{S \times S^n} \to C^0 = \oplus_i O_{D_i}$, whose components are given by restriction of sections. We set $G^0 := FM_{C^0}$. As explained in [Mea15, Sect. 6], the main steps in the computation of the formulas of [Sca09] and [Kru14] can be translated into the following statement

$$\hat{F}''' \hat{F}'' \cong \hat{F}'' \hat{F}'' \cong G'^R G'', \quad \hat{F}'' \hat{F}'' \cong G''R \hat{F}'' \cong G''R G''. \quad (5.9)$$

**Definition 5.5.1.** Let $Z$ be a smooth projective variety of arbitrary dimension $d$ and $n \geq 2$. The *truncated universal ideal functor* $G = FM_G : D^b(Z) \to D^b_{\mathcal{S}_n}(Z^n)$ is the Fourier–Mukai transform whose kernel is the complex

$$G := \mathcal{G}^* := (0 \to O_{Z \times Z^n} \to \bigoplus_{i=1}^{n} O_{D_i} \to 0) \in D^b_{\mathcal{S}_n}(Z \times Z^n).$$

Thus, there is the triangle of FM transforms $G \to G' \to G''$ with

$$G' := FM_{G'} = \hat{F}' = H^*(Z, -) \otimes O_{Z^n}, \quad G'' = FM_{G''} = \text{Inf}_{\mathcal{S}_{n-1}} p_n^* \text{triv}.$$ 

For $E \in D^b(Z)$ we have $G''(E) = \oplus_{i=1}^{n} p_i^* E$ (see also Subsection 5.2.8 for details about the inflation functor $\text{Inf}$ and its right adjoint $\text{Res}$). The right adjoints are

$$G' = FM_{G'^R} = H^*(Z^n, -) \otimes \omega_Z[d], \quad G'' = O_{Z^n} \otimes \omega_Z[d],$$

$$G'^R = FM_{G'^R} = [-] \otimes p_n^* \circ \text{Res}_{\mathcal{S}_{n-1}} p_n^* \otimes O_{D_i}.$$ 

In the surface case Equation (5.9) gives the following

**Lemma 5.5.2.** If $Z = S$ is a surface, $\hat{F}^R \hat{F} \cong G^R G$. \hfill $\square$

See [7, Sec. 5.5] for a further investigation of the relation between the functors $F$ and $G$. We compute the compositions of the kernels:

$$G'^R G' = (O_Z \otimes \omega_Z) \otimes S^n H^*(O_Z)[d],$$

$$G'^R G'' = (O_Z \otimes \omega_Z) \otimes S^n H^*(O_Z)[d],$$

$$G'^R G' = (O_Z \otimes \omega_Z) \otimes S^n H^*(O_Z),$$

$$G'^R G'' = O_{\Delta} \otimes S^{n-1} H^*(O_Z) \oplus (O_Z \otimes \omega_Z) \otimes S^{n-2} H^*(O_Z).$$
The induced map \( G^R G' \to G''^R G'' \) under these isomorphisms is given by evaluation as follows. Let \((e_i)_i\) be a basis of \( H^*(O_Z) = \text{Hom}(O_Z, O_Z[1]) \). Then the component
\[
(O_Z \boxtimes \omega_Z) \cdot e_{i_1} \cdots e_{i_n}[d] \to (O_Z \boxtimes \omega_Z) \cdot e_{i_1} \cdots e_{i_n}[d]
\]
is \( e_{i_k} \otimes \text{id}[d] \). The component
\[
(O_Z \boxtimes O_Z) \otimes S^{n-1} H^*(O_Z) \to (O_Z \boxtimes O_Z) \otimes S^{n-2} H^*(O_Z)
\]
of \( G''^R G' \to G''''^R G'''' \) is given in the same way and the component
\[
(O_Z \boxtimes O_Z) \otimes S^{n-1} H^*(O_Z) \to O_\Delta \otimes S^{n-1} H^*(O_Z)
\]
is the restriction map. The map \( G''^R G' \to G''''^R G'''' \) is given as follows. Let \((\vartheta_i)_i\) be the basis of \( H^*(O_Z)^\vee \) dual to \((e_i)_i\). Furthermore, let \( \vartheta_i \) correspond to the morphism \( \vartheta_i \in \text{Hom}(O_Z, \omega_Z[d-\ast]) \ni \text{Hom}(O_Z, O_Z[\ast])^\vee \) under Serre duality. Then the component
\[
(O_Z \boxtimes O_Z) \cdot e_{i_1} \cdots e_{i_k} \cdots e_{i_n} \to (O_Z \boxtimes \omega_Z) \cdot e_{i_1} \cdots e_{i_n}[d]
\]
is \( \vartheta_{i_k} \).

In the following we will use the commutative diagram
\[
\begin{array}{ccc}
G''^R G & \longrightarrow & G''''^R G'''' \\
\downarrow & & \downarrow \\
G'^R G' & \longrightarrow & G''''^R G'''' \\
\downarrow & & \downarrow \\
G^R G & \longrightarrow & G''''^R G''''
\end{array}
\] (5.10)

with exact triangles as columns and rows in order to deduce formulae for \( G^R G = FM_{G^R G} \).

5.5.1 The case of an even dimensional Calabi–Yau variety

If \( Z \) is a Calabi–Yau variety of even dimension \( d \), then \( H^*(O_Z) = \mathbb{C}[0] \oplus \mathbb{C}[-d] \) and \( S^k H^*(O_Z) = \mathbb{C}[0] \oplus \mathbb{C}[-d] \oplus \cdots \oplus \mathbb{C}[-d k] \). Let \( u \in H^d(O_Z) \) be the basis vector whose dual \( \vartheta \in \text{Hom}(O_Z, O_Z[d])^\vee \) corresponds to \( \text{id} \in \text{Hom}(O_Z, O_Z) \) under Serre duality. We denote the induced degree \( d \ell \) basis vector of \( S^k H^*(O_Z) \) by \( u' \).

**Lemma 5.5.3.** For a Calabi–Yau variety \( Z \) of even dimension \( d \) we have
\[
G^R G \cong \text{id} \oplus [-d] \oplus \cdots \oplus [-d(n-1)].
\]

**Proof.** By the description of the last subsection, the components \( O_{Z^2} \cdot u^\ell \to O_{Z^2} \cdot u^\ell \) of \( G''^R G' \to G''''^R G'''' \) and \( G'^R G' \to G''''^R G'''' \) equal the identity. By [Hub, Lem. 5], the top of the diagram (5.10) is isomorphic to
\[
\begin{array}{ccc}
G''^R G & \longrightarrow & O_{Z^2}[-d(n-1)] \longrightarrow O_\Delta([0] \oplus [-d] \oplus \cdots \oplus [-d(n-1)]) \\
\downarrow & & \downarrow \\
G'^R G & \longrightarrow & O_{Z^2}[-d(n-1)] \longrightarrow 0
\end{array}
\]
where the middle vertical map is the component $O_{Z^2} \cdot u^{n-1} \to O_{Z^2} u^n[d]$ of the morphism $G''_R G' \to G''_R G'$. By the above description it is the identity. Now the claim follows by the octahedral axiom. Alternatively, chase through long exact sequences of cohomology as in [Add16, Sect. 2.4].

**Theorem 5.5.4.** If $Z$ is a Calabi-Yau variety of even dimension, then the truncated universal ideal functor $G: D^b(Z) \to D_{\mathbb{P}^{n-1}}^b(Z^n)$ is a $\mathbb{P}^{n-1}$-functor.

**Proof.** By the previous lemma, condition (1) of the definition of a $\mathbb{P}^{n-1}$-functor is satisfied.

The proof that condition (2) is satisfied is analogous to the proof in the case of the non-truncated functor when $X$ is a K3 surface. Basically, one has to go through [Add16, Sect. 2.5] and replace $F$ by $G$, $q^*$ by $p_n^* \circ \text{triv} : D^b(Z) \to D_{\mathbb{P}^{n-1}}^b(Z^n)$ and its right adjoint $q_*$ by $(-)^{S_{n-1}} \circ p^*$, $g_s$ by $\text{Inf} : D_{\mathbb{P}^{n-1}}^b(Z^n) \to D_{\mathbb{P}^n}^b(Z^n)$ and its right adjoint $g^!$ by $\text{Res}, H^*(O_{\mathbb{P}^n})$ by $H^*(O_{Z^n})^S_{n-1}$, 2 by $d$, and the symplectic form $\sigma$, which occurs in [Add16, Sect. 2.5], by a generator of $H^d(O_Z)$.

Very roughly, the idea is the following: $G''_R G$ is identified with the direct summand $(p_n^* \circ \text{triv})^R((p_n^* \circ \text{triv})^R)^R = \text{id} \otimes H^*(Z^{n-1}, O_{Z^{n-1}}) \otimes H^*(O_{Z^n})$ and the monad structure of $(p_n^* \circ \text{triv})^R(p_n^* \circ \text{triv})$ is given by the cup product on $H^*(Z^{n-1}, O_{Z^{n-1}}) \otimes H^*(O_Z)$ which has the right form.

Condition (3) is easy to check since all occurring autoequivalences are simply shifts. □

**Remark 5.5.5.** If $Z$ is an odd dimensional Calabi-Yau variety, then $G$ fails to be spherical in an interesting way (compare Remark 5.3.7). Indeed, in this case $H^*(O_Z) = \mathbb{C}[0] \oplus \mathbb{C}[-d]$ and $S^k H^*(O_Z) = \mathbb{C}[0] \oplus \mathbb{C}[-d]$ for $k \geq 1$. The reason for the vanishing of the higher degrees is that the symmetric product is taken in the graded sense.

We can then check that for $n \geq 3$ we have $G''_R G \cong O_{\Delta}(\{0\} \oplus \{-d\})$ by using that the component $O_{Z^2} \otimes S^{n-1} H^*(O_Z) \to O_{Z^2} \otimes S^{n-2} H^*(O_Z)$ of the map $G''_R G' \to G''_R G''$ as well as the whole $G''_R G' \to G''_R G''$ are isomorphisms.

The above means that the cotwist of $G$ is an equivalence, but for dimension reasons the second axiom of a spherical functor cannot hold for $G$.

### 5.5.2 The case $H^*(O_Z) = \mathbb{C}[0]$

In this case also $S^k H^*(O_Z) = \mathbb{C}[0]$ for $k \geq 1$.

**Proposition 5.5.6.** The functor $G$ is fully faithful, i.e. $G''_R G = O_{\Delta}$.

**Proof.** The maps $G''_R G' \to G''_R G''$ and $G''_R G' \to G''_R G''$ are the identity on the components $O_{Z^2}$. Hence, $G''_R G \cong O_{\Delta} \otimes S^{n-1} H^*(O_Z)[-1]$ and $G''_R G = 0$. It follows that $G''_R G = O_{\Delta} \otimes S^{n-1} H^*(O_Z) = O_{\Delta}$. □

**Remark 5.5.7.** In particular, when $Z = S$ is a surface, the reader will recognize, using Lemma 5.5.2, the proof of Theorem 5.1.2.

### 5.5.3 The orthogonal complement of $\text{im } G$

Let, as in the previous subsection, $Z$ be a smooth projective variety such that the structure sheaf $O_Z$ is exceptional. We set $A := O_Z^\perp$ which means that we have the semi-orthogonal
decomposition $D^b(Z) = \langle A, O_Z \rangle$. For $A \in A$ we have $G'(A) = 0$ and thus $G(A) = \text{inf}(O_{Z^{n-1}} \boxtimes A)[1]$. Let $\varrho_n$ be the standard representation of $S_n$ which is given by the short exact sequence

$$0 \to \mathbb{C} \xrightarrow{\iota} \mathbb{C}^n \to \varrho_n \to 0$$

where $\iota : \mathbb{C} \to \mathbb{C}^n$ denotes the diagonal embedding of the trivial representation into the permutation representation. We have $G'(O_Z) = O_{Z^n}$ and $G''(O_Z) = O_{Z^n} \otimes \mathbb{C}^n$ from which $G(O_Z) = O_{Z^n} \otimes \varrho_n$ follows. We see that $\text{im}(G)$ equals the component $\langle B(n - 1, \mathbb{C}), O_{Z^n} \otimes \varrho_n \rangle$ of the semi-orthogonal decomposition described in Remark 5.4.7. Thus, we get a description of $\text{im}(G)^\perp$ and $\perp \text{im}(G)$. For example, in the case $n = 2$ we have $\text{im}(G)^\perp = \langle \omega_{Z^n}, B(0, \mathbb{C}) \rangle$ with

$$B(0, \mathbb{C}) = \langle \text{inf}(A_1 \boxtimes A_2) \mid A_1, A_2 \in A \rangle.$$

Let now $Z = X$ be an Enriques surface. By tensoring the exceptional sequence of Remark 5.4.9 by $E_{10}'$ we get the completely orthogonal exceptional sequence of line bundles $L_1, \ldots, L_9, O_X$ in $D^b(X)$. Using the above description of the image of $G$ one can check that $L_j^{E_{10}'} \in \text{im}(G)^\perp \cap \perp \text{im}(G) = \ker(G^R) \cap \ker(G^L)$ for $j = 1, \ldots, 9$. The objects $L_j^{E_{10}'}$ are in fact also contained in $\ker(F^R) \cap \ker(F^L)$ which proves Remark 5.3.14. Indeed, for $i = 2, \ldots, n$ the FM transform $\text{FM}_{C_{i-1}}$ is the composition

$$D^b(X) \xrightarrow{M_{a_i \text{triv}}} D^b_{E_{i} \times E_{n-i}}(X) \xrightarrow{p_{X, i}} D^b_{E_{i} \times E_{n-i}}(X \times X^{n-i}) \xrightarrow{\delta_i \times \text{id}} D^b_{E_{i} \times E_{n-i}}(X^i \times X^{n-i}) \xrightarrow{\text{inf}} D^b_{E_{i} \times E_{n-i}}(X^i \times X^{n-i})$$

where $a_i$ is the sign representation of $S_i$ and $\delta_i : X \to X^i$ is the diagonal embedding. The left adjoint $\text{FM}_{C_{i-1}}^L$ is

$$D^b(X) \xleftarrow{(-)^{S_i \times S_{n-i} \text{M}_a}} D^b_{E_{i} \times E_{n-i}}(X) \xleftarrow{p_{X, i}^!} D^b_{E_{i} \times E_{n-i}}(X \times X^{n-i}) \xleftarrow{(\delta_i \times \text{id})^*} D^b_{E_{i} \times E_{n-i}}(X^i \times X^{n-i}), \tag{5.11}$$

Now, let $\mathcal{L}$ be one of the $L_j$. Then $(\delta_i \times \text{id})^* \mathcal{L} = \mathcal{L}^{E_{i}^*} \boxtimes \mathcal{L}^{\varphi_{i}}$. The $E_i$-action on $\mathcal{L}^{E_{i}^*}$ is given by permuting the tensor factors. Since $\mathcal{L}$ is a line bundle, this action is trivial. Thus, after tensoring by $M_{a}$, the $E_i$-invariants vanish, hence $\text{FM}_{C_{i-1}}^L(\mathcal{L}^{\varphi_{i}}) = 0$. Since we already know that $G^L(L^{\varphi_{i}}) = 0$ we get $L^{\varphi_{i}} \in \ker(F^L)$. The functor $\text{FM}_{C_{i-1}}^R$ is given by the composition $(5.11)$ with $(\delta_i \times \text{id})^*$ replaced by $(\delta_i \times \text{id})^!$ and $p_{X, i}^?$ replaced by $p_{X, i}$. Thus, also $\text{FM}_{C_{i-1}}^R(\mathcal{L}^{\varphi_{i}}) = 0$ and hence $L^{\varphi_{i}} \in \ker(F^R)$.

References


Chapter 6

On derived autoequivalences of Hilbert schemes and generalized Kummer varieties

Abstract

We show that for every smooth projective surface $X$ and every $n \geq 2$ the push-forward along the diagonal embedding gives a $\mathbb{P}^{n-1}$-functor into the $\mathfrak{S}_n$-equivariant derived category of $X^n$. Using the Bridgeland–King–Reid–Haiman equivalence this yields a new autoequivalence of the derived category of the Hilbert scheme of $n$ points on $X$. If the canonical bundle of $X$ is trivial and $n = 2$ this autoequivalence coincides with the known EZ-spherical twist induced by the boundary of the Hilbert scheme. We also find $n^4$ orthogonal $\mathbb{P}^{n-1}$-objects on the generalised Kummer variety associated to an abelian surface. They generalise the 16 spherical objects on the Kummer surface given by the exceptional curves.

6.1 Introduction

The bounded derived category of coherent sheaves is an important homological invariant of a smooth projective variety. In particular, every equivalence between these categories induces isomorphisms on the level of the Grothendieck groups as well as on Hochschild homology and cohomology. If the canonical bundle of the variety is ample or anti-ample, all autoequivalences of the derived category are standard by a theorem of Bondal and Orlov [BO01]. That means that the group of autoequivalences is generated by push-forwards along automorphisms, tensor products by line bundles, and degree shifts. Besides that, there are only very few results giving explicit descriptions of the groups of autoequivalences. These are due to Orlov [Orl02] for abelian varieties, Broomhead and Ploog [BP14] for toric surfaces, and Bayer and Bridgeland [BB17] for K3 surfaces of Picard rank one. For other varieties, the current focus is on constructing and studying new autoequivalences which might eventually lead to descriptions of the full group of autoequivalences. In this paper, we consider the case of the Hilbert scheme $X^{[n]}$ of $n$ points on a smooth projective surface $X$. 
The derived McKay correspondence of Bridgeland, King, and Reid [BKR01] gives, under some assumptions, a derived equivalence between a global quotient orbifold and a certain crepant resolution. It can be seen as a categorical version of the general philosophy inspired from physics that the geometry of a global quotient orbifold should be the same as the geometry of a crepant resolution; compare the conjectures of Ruan [Rua06]. Haiman [Hai01] showed that the derived McKay correspondence applies to the Hilbert schemes of points on a surface to give an equivalence $\mathbb{D}^b(X^{[n]}) \xrightarrow{\sim} \mathbb{D}^b_{\mathcal{S}_n}(X^n)$ to the $\mathcal{G}_n$-equivariant derived category of the cartesian product of $X$.

In [Plo07], Ploog used this to give a general construction which associates to autoequivalences on the surface autoequivalences on the Hilbert scheme; see Section 6.5 for details. More recently, Ploog and Sosna [PS14] gave a construction that takes spherical objects (see Seidel and Thomas [ST01]) on the surface and produces $\mathbb{P}^n$-objects (see Huybrechts and Thomas [HT06]) on $X^{[n]}$. These $\mathbb{P}^n$-objects in turn induce further derived autoequivalences. Apart from these constructions, there are few known autoequivalences of $\mathbb{D}^b(X^{[n]})$: 

- The group of characters $\hat{\mathcal{S}}_n \simeq \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{D}^b_{\mathcal{S}_n}(X^n)$ hence also on $\mathbb{D}^b(X^{[n]})$.

- Addington introduced in [Add16] the notion of a $\mathbb{P}^n$-functor generalising the $\mathbb{P}^n$-objects of Huybrechts and Thomas; see Section 6.2 for details. He showed that for $X$ a K3 surface and $n \geq 2$ the Fourier–Mukai transform $\mathbb{D}^b(X) \to \mathbb{D}^b(X^{[n]})$ induced by the universal sheaf is a $\mathbb{P}^{n-1}$-functor. This yields an autoequivalence of $\mathbb{D}^b(X^{[n]})$ for every K3 surface $X$ and every $n \geq 2$.

- For $X = A$ an abelian surface, the pull-back along the summation map $\Sigma: A^{[n]} \to A$ is a $\mathbb{P}^{n-1}$-functor and thus induces an autoequivalence; see Meachan [Mea15, Theorem 5.2].

- Let the canonical bundle of $X$ be trivial and $n = 2$. Then every line bundle on the exceptional divisor of the Hilbert–Chow morphism $X^{[2]} \to X^2/\mathcal{G}_2$ is an EZ-spherical object in the sense of Horja [Hor05] and thus also induces an autoequivalence; see [Huy06, Example 8.49 (iv)].

In this article we generalise this last example to surfaces with arbitrary canonical bundle and to arbitrary $n \geq 2$. More precisely, we consider the functor $F: \mathbb{D}^b(X) \to \mathbb{D}^b_{\mathcal{S}_n}(X^n)$ which is defined as the composition of the functor $\text{triv}: \mathbb{D}^b(X) \to \mathbb{D}^b_{\mathcal{S}_n}(X)$ given by equipping every object with the trivial $\mathcal{G}_n$-linearisation and the push-forward $\delta_+: \mathbb{D}^b_{\mathcal{S}_n}(X) \to \mathbb{D}^b_{\mathcal{G}_n}(X^n)$ along the diagonal embedding. Then we show in Section 6.3 the following.

**Theorem 6.1.** For every $n \geq 2$ and every smooth projective surface $X$, the functor $F: \mathbb{D}^b(X) \to \mathbb{D}^b_{\mathcal{S}_n}(X^n)$ is a $\mathbb{P}^{n-1}$-functor.

In Section 6.4 we show that for $n = 2$ the induced autoequivalence coincides under the derived McKay correspondence $\mathbb{D}^b_{\mathcal{G}_2}(X^2) \simeq \mathbb{D}^b(X^{[2]})$ with the autoequivalence induced by a line bundle on the exceptional divisor. In Section 6.5 we compare the autoequivalence induced by $F$ to some other derived autoequivalences of $X^{[n]}$ showing that it differs essentially from the standard autoequivalences and, in the K3 surface case, from the autoequivalence induced by Addington’s $\mathbb{P}$-functor. In particular, the Hilbert scheme always has non-standard autoequivalences even if $X$ is a Fano surface. In the last section we consider the case that $X = A$ is an abelian surface. We find $n^4$ pairwise orthogonal $\mathbb{P}^{n-1}$-objects on the generalised...
Kummer variety $K_{n-1} A \subset A^{[n]}$ which arise by restricting our $\mathbb{P}^{n-1}$-functor $F$ in an appropriate way. They generalise the 16 spherical objects on the Kummer surface given by the line bundles $\mathcal{O}_C(-2)$ on the exceptional curves.

**Conventions.** In this paper, all varieties are smooth and projective and defined over $\mathbb{C}$. The *derived category* of a variety $X$ is the bounded derived category $\mathcal{D}^b(X) := \mathcal{D}^b(\text{Coh}(X))$ of coherent sheaves; see e.g. [Huy06] for details. If a finite group $G$ acts on $X$, the *equivariant derived category* is the bounded derived category $\mathcal{D}^b_G(X) := \mathcal{D}^b(\text{Coh}_G(X))$ of $G$-equivariant coherent sheaves; see [BKR01, Section 4] for details. We do not distinguish in the notation between functors of abelian categories and their derived functors. Given a line bundle $L$ on a variety $X$ we write $M_L := (\_)_\otimes L$ for the tensor product functor. For a complex $E$ we denote its $q$-th cohomology by $H^q(E)$ and set $H^*(E) := \oplus_{i \in \mathbb{Z}} H^i(E)[-i]$.

**Acknowledgements.** The author wants to thank Daniel Huybrechts and Ciaran Meachan for helpful discussions. This work was supported by the SFB/TR 45 of the DFG (German Research Foundation). As communicated to the author shortly before he posted this article on the ArXiv, Will Donovan independently discovered the $\mathbb{P}^n$-functor $F$. The author also thanks Nicolas Addington and Will Donovan for pointing out a mistake in the first version of the paper and the referees for very helpful comments.

### 6.2 $\mathbb{P}^n$-functors

In this section, we introduce $\mathbb{P}$-functors together with some of their basic properties.

Let $G$ and $H$ be finite groups acting on a varieties $M$ and $N$, respectively. Then for every object $P \in \mathcal{D}^b_{G \times H}(M \times N)$ there is the associated *equivariant Fourier–Mukai transform*

$$
\text{FM}_P = \text{pr}_{N*}(P \otimes \text{pr}_M^*)^{G \times 1}: \mathcal{D}^b_G(M) \to \mathcal{D}^b_H(N);
$$

(6.1)

see [Plo07, Section 1.2] for details. Since we assume that $M$ and $N$ are smooth and projective, there are left and right adjoint Fourier–Mukai transforms. Following [Add16, Section 3.1], we define a $\mathbb{P}^n$-functor as an equivariant Fourier–Mukai transform $F: \mathcal{D}^b_G(M) \to \mathcal{D}^b_H(N)$ with left and right adjoints $L, R: \mathcal{D}^b_H(N) \to \mathcal{D}^b_G(M)$ such that

1. There is an autoequivalence $D$ of $\mathcal{D}^b_G(M)$ (called the $\mathbb{P}$-cotwist of $F$) such that

$$
RF \simeq \text{id} \oplus D \oplus D^2 \oplus \cdots \oplus D^n.
$$

(6.2)

2. Let $j: D \hookrightarrow RF$ the inclusion of the direct summand in (6.2). The pieces $c_{ij}: D^i \to D^j$ of the natural transformation

$$
D \oplus D^2 \oplus \cdots \oplus D^n \oplus D^{n+1} \simeq DRF \xrightarrow{jRF} RFRF \xrightarrow{ReF} RF \simeq \text{id} \oplus D \oplus \cdots \oplus D^n
$$

(6.3)

are isomorphisms for $i = j$ and zero for $i < j$. That means that (6.3) written as a matrix is of the form

$$
\begin{pmatrix}
* & * & \cdots & * & * \\
1 & * & \cdots & * & * \\
0 & 1 & \cdots & * & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & *
\end{pmatrix},
$$

(6.4)
3. $R \simeq D^n L$. If $D_G^b(M)$ and $D_H^b(N)$ have Serre functors $S_M$ and $S_N$, this is equivalent to $S_NFD^n \simeq FS_M$.

The $\mathbb{P}$-twist associated to a $\mathbb{P}^n$-functor $F$ is defined as the double cone

$$PF = \text{cone}(\text{cone}(FDR \to FR) \to \text{id}) .$$

The map defining the inner cone is given by the composition

$$FDR \xrightarrow{FjR} FR \xrightarrow{\varepsilon FR} FR .$$

The map defining the outer cone is induced by the counit $\varepsilon := \varepsilon_F: FR \to \text{id}$; for details see [Add16, Section 3.3]. Taking the cones of the Fourier–Mukai transforms indeed makes sense, since all the occurring maps are induced by maps between the Fourier–Mukai kernels; see [AL12]. Alternatively, instead of restricting to FM transforms one can work with dg enhancements; compare [AL13].

We set $\ker R := \{ B \in D_H^b(N) \mid RB = 0 \}$. By the adjoint property it equals the right-orthogonal complement $(\text{im } F)^\perp := \{ B \in D_H^b(N) \mid \text{Hom}_{D_H^b(N)}(F(A), B) = 0 \text{ for all } A \in D_G^b(M) \}$.

**Proposition 6.2.1** ([Add16, Proposition 3.3 & Theorem 3]). Let $F: D_G^b(M) \to D_H^b(N)$ be a $\mathbb{P}^n$-functor.

1. We have $P_F(B) \simeq B$ for $B \in \ker R$.
2. $P_F F \simeq FD^{n+1}[2]$.
3. The objects in $\text{im } F \cup \ker R$ form a spanning class of $D_H^b(N)$.
4. $P_F$ is an autoequivalence of $D_H^b(N)$.

**Example 6.2.2.** Let $N$ be a smooth projective variety. An (equivariant) $\mathbb{P}^n$-object is an object $E \in D_H^b(N)$ such that $E \otimes \omega_N \simeq E$ and

$$\text{Hom}_{D_H^b(N)}^*(E, E) = \text{Hom}_{D_H^b(N)}^*(E, E)^H \simeq H^*(\mathbb{P}^n_\mathbb{C}, \mathbb{C})$$

as $\mathbb{C}$-algebras (the ring structure on the left-hand side is the Yoneda product and on the right-hand side the cup product); see [HT06]. A $\mathbb{P}^n$-object can be identified with the $\mathbb{P}^n$-functor

$$F: D^b(\text{pt}) \to D_H^b(N) , \quad \mathbb{C} \mapsto E$$

with $\mathbb{P}$-cotwist $D = [-2]$. Note that the right adjoint of $F$ is given by $R = \text{Hom}_{D^b_H(N)}^*(E, )$. The $\mathbb{P}^n$-twist associated to the functor $F$ is the same as the $\mathbb{P}^n$-twist associated to the object $E$ as defined by Huybrechts and Thomas [HT06].

**Example 6.2.3.** A $\mathbb{P}^1$-functor is the same as a spherical functor (see [Rou06, Section 8.1], [Add16, Section 1], [AL13]) with the property that the exact triangle $\text{id} \xrightarrow{2} RF \to D$, with $\eta = \eta_F$ being the adjunction unit, splits. In this case there is also the spherical twist given by $T_F := \text{cone}(FR \xrightarrow{\varepsilon} \text{id})$. It is an autoequivalence satisfying $T_F^2 \simeq P_F$; see [Add16, Section 3.2].

**Lemma 6.2.4.** Let $F: D_G^b(M) \to D_H^b(N)$ be a $\mathbb{P}^n$-functor.
1. Let $\Psi: D^b_G(M') \to D^b_G(M)$ be a Fourier–Mukai equivalence. Then $F \Psi$ is again a $\mathbb{P}^n$-functor. The associated twist is given by $P_{F, \Psi} \simeq P_F$.

2. Let $\Phi: D^b_H(N) \to D^b_H(N')$ be a Fourier–Mukai equivalence. Then $\Phi F$ is again a $\mathbb{P}^n$-functor. The associated twist is given by $P_{\Phi, F} \simeq \Phi P_F \Phi^{-1}$.

Proof. Let $D$ be the $\mathbb{P}$-cotwist of $F$ and fix an isomorphism $k: RF \simeq \id \oplus D \oplus \cdots \oplus D^n$ with components $k_i: RF \to D^i$. Set $\hat{D} := \Psi^{-1} D \Psi$. We consider the isomorphisms $\ell_i = \eta_i^{-1}: \Psi^{-1} \Psi \xrightarrow{\simeq} \hat{D}$, $\ell_1 = \id$, and

$$\ell_i = \Psi^{-1} D \eta_{i-1} D \cdots D \eta_0 D \Psi: \Psi^{-1} D^i \Psi \xrightarrow{\simeq} \hat{D}^i \text{ for } i \geq 2.$$ 

The functor $\hat{F} := F \Psi$ has adjoints $\hat{L} := \Psi^{-1} L$ and $\hat{R} := \Psi^{-1} R$. There is the isomorphism

$$k: \hat{R} \hat{F} \xrightarrow{\Psi^{-1} k} \Psi^{-1} (\id \oplus D \oplus \cdots \oplus D^n) \Psi \xrightarrow{\oplus \ell_i} \id \oplus \hat{D} \oplus \cdots \oplus \hat{D}^n$$

whose components we denote by $\hat{k}_i: \hat{R} \hat{F} \to \hat{D}^i$. In order to verify condition (ii) of a $\mathbb{P}^n$-functor for $\hat{F}$, we need to show that the composition

$$\hat{c}_{i+1, j}: \hat{D} \hat{D}^i \xrightarrow{\hat{k}_i^{-1} \hat{k}_j^{-1}} \hat{R} \hat{F} \hat{R} \hat{F} \xrightarrow{\hat{R} \hat{F} \hat{R} \hat{F}} \hat{R} \hat{F} \xrightarrow{\hat{k}_j} \hat{D}^j$$

(6.5)

is zero for $j > i + 1$ and an isomorphism for $j = i + 1$. We have $\varepsilon_{\hat{F}} = \varepsilon_F \circ F \varepsilon_{\Psi} R$ and the diagram

$$\begin{array}{ccc}
D \Psi \Psi^{-1} D^i & \xrightarrow{\Psi^{-1} D \eta_{i-1} \Psi} & RF \Psi^{-1} RF \\
D \varepsilon_{\Psi} D^i & \downarrow & \downarrow RF \varepsilon_{\Psi} RF \\
\hat{D} \hat{D}^i & \xrightarrow{\hat{k}_i^{-1} \hat{k}_j^{-1}} & FRFR
\end{array}$$

commutes. Thus, (6.5) can be rewritten as

$$\hat{D} \hat{D}^i \xrightarrow{\ell_i^{-1} \hat{k}_i^{-1}} \Psi^{-1} D (\Psi \Psi^{-1} D^i) \Psi \xrightarrow{\Psi^{-1} D \varepsilon_{\Psi} D^i \Psi} \Psi^{-1} D^i \Psi \xrightarrow{\Psi^{-1} (\varepsilon_F R \varepsilon_{\Psi} F k_i^{-1} \varepsilon_{\Psi})} \Psi^{-1} D^j \Psi \xrightarrow{\ell_j} \hat{D}^j.$$ 

(6.6)

We set $c_{i+1, j} = \varepsilon_F R \varepsilon_{\Psi} F \circ \hat{k}_i^{-1} D \hat{D}^i \to D^j$ so that the third arrow of (6.6) is given by $\Psi^{-1} c_{i+1, j} \Psi$. Since $F$ is a $\mathbb{P}^n$-functor, the morphism $c_{i+1, j}$ is zero for $j > i + 1$ and an isomorphism for $j = i + 1$. Thus, the whole composition (6.6) is zero for $j > i + 1$. Furthermore, since $\varepsilon_F$ and the $\ell_i$ are isomorphisms as well, (6.6) is an isomorphism for $j = i + 1$. Finally, $\hat{R} \simeq \Psi^{-1} R \simeq \Psi^{-1} D^n L \simeq \Psi^{-1} D^n \Psi^{-1} L \simeq \hat{D}^n L$ which is condition (iii) of $\hat{F}$ being a $\mathbb{P}^n$-functor. One can check that all squares of the diagram

$$\begin{array}{ccc}
\hat{F} \hat{D} \hat{R} & \xrightarrow{\hat{F} \hat{k}_i^{-1} \hat{R}} & \hat{F} \hat{R} \hat{F} \hat{R} \\
F \varepsilon_{\Psi} D \varepsilon_{\Psi} R & \downarrow & \downarrow F \varepsilon_{\Psi} R \varepsilon_{\Psi} R \\
FRFR & \xrightarrow{\varepsilon_F FR \varepsilon_{\Psi} FR} & FRFR
\end{array}$$

commute. Since all the vertical maps are isomorphisms, we get an induced isomorphism of the double cones

$$P_{\hat{F}} \simeq \cone(\cone(\hat{F} \hat{D} \hat{R} \to \hat{F} \hat{R}) \to \id) \simeq \cone(\cone(FDR \to FR) \to \id) \simeq P_F.$$ 

The proof of part (ii) of the lemma is similar; compare [ST01, Lemma 2.11].
We say that two objects $E, F \in D^b_H(N)$ are orthogonal if $\Hom^*_b(D^b_H(N), E, F) = 0 = \Hom^*_b(D^b_H(N), F, E)$. Note that if $E$ is a $\mathbb{P}^n$-object, we have in particular $E \otimes \omega_N \simeq E$. Thus, by Serre duality, the vanishing of $\Hom^*_b(D^b_H(N), E, F)$ implies the vanishing of $\Hom^*_b(D^b_H(N), F, E)$ and vice versa.

**Corollary 6.2.5.** Let $E_1, \ldots, E_n \in D^b_H(N)$ be pairwise orthogonal $\mathbb{P}^n$-objects with twists $p_i := P_{E_i}$. Then

$$\gamma: \mathbb{Z}^n \to \text{Aut}(D^b_H(N)) \quad , \quad (\lambda_1, \ldots, \lambda_n) \mapsto p_1^{\lambda_1} \cdots p_n^{\lambda_n}$$

defines a group isomorphism $\mathbb{Z}^n \simeq (p_1, \ldots, p_n) \subset \text{Aut}(D^b_H(M))$.

**Proof.** By Proposition 6.2.1(i) we have $p_i(E_j) = E_j$ for $i \neq j$. Thus, the $p_i$ commute by Lemma 6.2.4(ii) which means that the map $\gamma$ is indeed a group homomorphism onto the subgroup generated by the $p_i$. Let $g = p_1^{\lambda_1} \cdots p_n^{\lambda_n}$. Then $g(E_i) = E_i[2n\lambda_i]$ by Proposition 6.2.1. Thus, $g = \text{id}$ implies $\lambda_1 = \cdots = \lambda_n = 0$. \hfill $\Box$

**Lemma 6.2.6 ([Add16, Proposition 1.2]).** Let $P$ be an autoequivalence of $D^b_H(N)$ and $A, B \in D^b_H(N)$ objects such that $P(A) = A[i]$ and $P(B) = B[j]$ for some $i \neq j \in \mathbb{Z}$. Then $A$ and $B$ are orthogonal.

**Remark 6.2.7.** Let $F$ be a $\mathbb{P}^n$-functor with $\mathbb{P}$-cotwist $D = [-\ell]$ for some $\ell \in \mathbb{Z}$. Then there does not exist a non-zero object $A$ with $P_F(A) = A[i]$ for any values of $i$ besides $0$ and $-n\ell + 2$. Indeed, by Proposition 6.2.1 together with Lemma 6.2.6 such an object would be orthogonal to the spanning class in $F \cup \ker R$.

### 6.3 The diagonal embedding

In this section, we will prove Theorem 6.1.1. Let $X$ be a smooth projective surface over $\mathbb{C}$ and $2 \leq n \in \mathbb{N}$. We denote by $\delta: X \to X^n$ the diagonal embedding. We want to show that the composition

$$F: D^b(X) \xrightarrow{\text{triv}} D^b_{\mathcal{S}_n}(X) \xrightarrow{\delta^*} D^b_{\mathcal{S}_n}(X^n)$$

is a $\mathbb{P}^{n-1}$-functor. Here, $\mathcal{S}_n$ is considered to act trivially on $X$ and $\text{triv}: D^b(X) \to D^b_{\mathcal{S}_n}(X)$ is the functor which equips each object with the trivial linearisation. The right adjoint of $F$ is given by the composition

$$R: D^b_{\mathcal{S}_n}(X^n) \xrightarrow{\delta^*} D^b_{\mathcal{S}_n}(X) \xrightarrow{(\cup)_{\mathcal{S}_n}} D^b(X)$$

of the usual right adjoint of the push-forward (see [LH09, Proposition 28.8] for equivariant Grothendieck duality for regular embeddings) and the functor of taking invariants. The approach is to compute the composition $\delta^* \delta_*$ first. Then taking $\mathcal{S}_n$-invariants gives the composition $RF$.

We consider the standard representation $\rho$ of $\mathcal{S}_n$ as the quotient of the permutation representation $\mathbb{C}^n$ by the one-dimensional invariant subspace. The normal bundle sequence

$$0 \to T_X \to T_{X^n}\vert_X \to N \to 0,$$

where $N := N_\delta = N_{X/X^n}$, is of the form

$$0 \to T_X \to T^{\mathbb{C}^n}_X \to N \to 0.$$
where the map $T_X \to T_X^{\oplus n}$ is the diagonal embedding. When considering $T_X|_X$ as a $\mathcal{G}_n$-sheaf equipped with the natural linearisation it is given by $T_X \otimes \mathbb{C}^n$ where $\mathbb{C}^n$ is the permutation representation. Thus, as a $\mathcal{G}_n$-sheaf, the normal bundle $N$ equals $T_X \otimes \varrho$. We also see that the normal bundle sequence splits using e.g. the splitting

$$T_X \otimes \mathbb{C}^n \to T_X, \quad (v_1, \ldots, v_n) \mapsto \frac{1}{n}(v_1 + \cdots + v_n).$$

**Theorem 6.3.1** ([AC12, Theorem 1.4]). Let $\iota : Z \hookrightarrow M$ be a regular embedding of codimension $c$ such that the normal bundle sequence splits. Then there is an isomorphism

$$\iota^*\iota_*(\_ \otimes (\bigoplus_{i=0}^{c} \wedge^i N^\vee_{Z/M}[i]))$$

of endofunctors of $D^b(Z)$.

Recall that the right-adjoint $\iota^!$ of $\iota_*$ is given by $\iota^!(\_ \otimes \iota^! \mathcal{O}_M$ and $\iota^! \mathcal{O}_M = \wedge^c_{\text{conderived}} N[-c]$; see [Har66, Corollary III.7.3].

**Corollary 6.3.2.** Under the same assumptions, there is an isomorphism $\iota^!\iota_!(\_) \simeq (\_ \otimes (\bigoplus_{i=0}^{c} \wedge^i N^\vee_{Z/M}[-i])).$

**Proof.** Apply the tensor product with $\iota^! \mathcal{O}_M \simeq \wedge^c N^\vee_{Z/M}[-c]$ on both sides of (6.7). \qed

In the case that $\iota = \delta$ from above this yields the isomorphism of functors

$$\delta^!\delta_*(\_ \otimes (\bigoplus_{i=0}^{2(n-1)} \wedge^i (T_X \otimes \varrho)[-i])).$$

(6.8)

**Lemma 6.3.3.** Let $\iota : Z \hookrightarrow M$ be as in Theorem 6.3.1 and let $\varepsilon := \varepsilon_{\iota*, \iota^!} : \iota_* \iota^! \to \text{id}$ be the counit of adjunction. For $0 \leq i, j \leq c$ with $i + j \leq c$, the component

$$(\_ \otimes \wedge^i N^\vee_{Z/M} \otimes \wedge^j N^\vee_{Z/M}[-(i + j)]) \to (\_ \otimes \wedge^{i+j} N^\vee_{Z/M}[-(i + j)])$$

of the morphism $\iota^!\varepsilon_{\iota*} : \iota^!\iota_* \to \iota_* \iota^!$ is given by the wedge pairing.

**Proof.** For $E \in D^b(M)$ the object $\iota^! E$ can be identified with $\text{Hom}_M(\iota_* \mathcal{O}_Z, E)$, the latter considered as an object in $D^b(Z)$. Under this identification the counit map $\varepsilon : \text{Hom}_M(\iota_* \mathcal{O}_Z, E) \to E$ is given by the evaluation $\varphi \mapsto \varphi(1)$; see [Har66, Section III.6]. Corollary 6.3.2 says in particular that $\iota^!\iota_* B \simeq B \otimes_{\mathcal{O}_Z} \iota^!\iota_* \mathcal{O}_Z$ for $B \in D^b(Z)$. This gives the identifications $\iota^!\iota_* B \simeq \text{Hom}_M(\iota_* \mathcal{O}_Z, \iota_* \mathcal{O}_Z) \otimes_{\mathcal{O}_Z} B$ and

$$\iota^!\iota_* \iota^!\iota_* B \simeq \text{Hom}_M(\iota_* \mathcal{O}_Z, \iota_* \mathcal{O}_Z) \otimes_{\mathcal{O}_Z} \text{Hom}_M(\iota_* \mathcal{O}_Z, \iota_* \mathcal{O}_Z) \otimes_{\mathcal{O}_Z} B.$$

Under these identifications, the component

$$\text{Ext}^i_M(\iota_* \mathcal{O}_Z, \iota_* \mathcal{O}_Z) \otimes_{\mathcal{O}_Z} \text{Ext}^j_M(\iota_* \mathcal{O}_Z, \iota_* \mathcal{O}_Z) \otimes_{\mathcal{O}_Z} B \to \text{Ext}^{i+j}_M(\iota_* \mathcal{O}_Z, \iota_* \mathcal{O}_Z) \otimes_{\mathcal{O}_Z} B$$

of the monad multiplication equals the Yoneda product. The Yoneda product corresponds to the wedge product under the isomorphism $\text{Ext}^i_M(\mathcal{O}_Z, \mathcal{O}_Z) \simeq \wedge^i N^\vee_{Z/M}$; see [LH09, Proposition 28.8(3)]. \qed

157
Corollary 6.3.5. For a vector bundle $E$ on $X$ of rank two and $0 \leq \ell \leq n-1$ there is an isomorphism of line bundles $[\wedge^{2\ell}(E \otimes \mathcal{O})] \cong (\wedge^2 E)^{\otimes \ell}$.

Proof. The isomorphism is given by composing the morphism

$$(\wedge^2 E)^{\otimes \ell} \to \wedge^{2\ell}(E \otimes \mathcal{O}^n), \quad x_1 \otimes \cdots \otimes x_{\ell} \mapsto \sum_{1 \leq i_1 < \cdots < i_\ell \leq n} x_{i_1}e_{i_1} \wedge \cdots \wedge x_{i_\ell}e_{i_\ell}$$

with the projection induced by the quotient $\mathbb{C}^n \to \mathcal{O}$.

We set $D := (\underline{\wedge} \otimes \wedge^2 T_X[-2] \cong M_{\omega_X^-}[-2] \cong S_X^{-1}$ as the inverse of the Serre functor on $X$.

Corollary 6.3.6. There is the isomorphism of functors $RF \cong \text{id} \oplus D \oplus D^2 \oplus \cdots \oplus D^{n-1}$.

Proof. We have $RF = (\underline{\wedge})^{\otimes n} \delta^i \delta^j \text{triv}$. The assertion follows by formula (6.8) and Corollary 6.3.5. □

Lemma 6.3.7. The functor $F$ fulfils condition (ii) of a $\mathbb{P}^{n-1}$-functor with cotwist $D = S_X^{-1}$.

Proof. All the components $c_{ij} : DD^{i-1} = D^i \to D^j$ of the morphism (6.3) are induced by morphisms between the Fourier–Mukai kernels. The FM kernel of $D^i = S_X^{-i}$ is $\iota_* \omega_X^{-i}[-2i]$ with $\iota : X \to X \times X$ being the diagonal embedding. For $i < j$ we have

$$\text{Hom}(\iota_* \omega_X^{-i}[-2i], \iota_* \omega_X^{-j}[-2j]) = \text{Ext}^2(i-j)(\iota_* \omega_X^{-i}, \iota_* \omega_X^{-j}) = 0$$

hence $c_{ij} = 0$. The generators $\omega^{\ell}$ from Lemma 6.3.4 are mapped to each other by the wedge product. For $i = 1, \ldots, n-1$ the components $c_{ii} : DD^{i-1} \to D^i$ are given by the wedge product; see Lemma 6.3.3. Hence, they are isomorphisms. □

Lemma 6.3.8. There is an isomorphism $S_X F D^{n-1} = FS_X$.

Proof. For $\mathcal{E} \in D^0(X)$ there are natural isomorphisms

$$S_X F D^{n-1} = S_X \omega_X^{n} \otimes \delta(\mathcal{E} \otimes \omega_X^{-(n-1)}[-2(n-1)]) \cong S_X \delta^i \delta^j \text{triv}.$$

where the second-to-last isomorphism is the projection formula. □

All this together shows Theorem 6.1.1. This means that $F = \delta \text{triv}$ is indeed a $\mathbb{P}^{n-1}$-functor with $\mathbb{P}$-cotwist $D = S_X^{-1}$.
6.4 Composition with the Bridgeland–King–Reid–Haiman equivalence

Let $F: D^b(X) \to D^b_{\mathbb{E}_n}(X^n)$ be the $\mathbb{P}^{n-1}$-functor of the previous section and let $\Phi: D^b(X^{[n]}) \to D^b_{\mathbb{E}_n}(X^n)$ be the derived McKay correspondence as defined below. By Lemma 6.2.4, the composition $\Phi^{-1}F: D^b(X) \to D^b(X^{[n]})$ is again a $\mathbb{P}^{n-1}$-functor and thus yields an autoequivalence of the derived category of the Hilbert scheme. In this section we explicitly compute $\Phi^{-1}F$ in the case $n = 2$.

The isospectral Hilbert scheme $I^nX \subset X^{[n]} \times X^n$ is the reduced fibre product $I^nX := (X^{[n]} \times S^nX)_{red}$ with the defining morphisms being the Hilbert–Chow morphism $\mu: X^{[n]} \to S^nX$ and the quotient morphism $\pi: X^n \to S^nX := X^n/\mathbb{E}_n$. Thus, there is the commutative diagram

$$
\begin{array}{ccc}
I^nX & \xrightarrow{p} & X^n \\
\downarrow q & & \downarrow \pi \\
X^{[n]} & \xrightarrow{\mu} & S^nX.
\end{array}
$$

The Bridgeland–King–Reid–Haiman equivalence (or derived McKay correspondence) is the functor

$$
\Phi := FM_{\mathcal{O}_{I^nX}} \simeq p_*q^*\text{triv}: D^b(X^{[n]}) \to D^b_{\mathbb{E}_n}(X^n).
$$

By [BKR01, Theorem 1.1] and [Hai01, Theorem 5.1] it is indeed an equivalence (note that the statement of Theorem 5.1 of [Hai01] is for $X = \mathbb{A}^2$ but remains true for arbitrary smooth quasi-projective surfaces as pointed out in [Hai01, Section 5.1] and [Sca09a, Section 1]). The isospectral Hilbert scheme can be identified with the blow-up of $X^n$ along the centre $\{ (x_1, \ldots, x_n) \in X^n \mid x_i = x_j \text{ for some } i \neq j \}$ (see [Hai01, Proposition 3.4.2 & Corollary 3.8.3]). Thus, $I^2X$ equals the blow-up of $X^2$ in the diagonal $\Delta$. In particular, $I^2X$ is smooth. We denote by

$$
j: I^2X \hookrightarrow X^{[2]} \times X^2
$$

the closed embedding, by $E \subset I^2X$ the exceptional divisor of the blow up $p: I^2X \to X^2$, and by $\tau = (1, 2)$ the non-trivial element of $\mathbb{S}_2$.

**Lemma 6.4.1.** The functor $\Phi^{-1}: D^b_{\mathbb{S}_2}(X^2) \to D^b(X^{[2]})$ is given by the equivariant Fourier–Mukai transform $FM_{Q}$ with kernel $Q = j_*\mathcal{O}(E) \in D^b_{\mathbb{S}_2}(X^2 \times X^{[2]})$. The $\mathbb{S}_2$-linearisation of $Q$ restricts on $E$ to $\tau$ acting by $-1$ on $\mathcal{O}_E(E)$.

**Proof.** By the general formula for the right-adjoint of a Fourier–Mukai transform (see e.g. [Huy06, Proposition 5.9]), we have $Q = O_{I^2X}^{[2]} \otimes pr_{X^{[2]}[2]}^* \omega_{X^{[2]}[4]}$ where $pr_{X^{[2]}[2]}: X^{[2]} \times X^2 \to X^{[2]}$ denotes the projection. The canonical bundle of the blow-up is given by $\omega_{I^2X} \simeq p^*\omega_{X^2} \otimes \mathcal{O}(E)$. Let $N$ be the normal bundle of the codimension 4 regular embedding $j: I^2X \hookrightarrow X^{[2]} \times X^2$. By the adjunction formula

$$
\wedge^4N \simeq j^*\omega_X^{[2]} \otimes \omega_{I^2X} \simeq q^*\omega_X^{[2]} \otimes \mathcal{O}(E).
$$

It follows by Grothendieck duality for regular embeddings that

$$
Q \simeq O_{I^2X}^{[2]} \otimes pr_{X^{[2]}[2]}^* \omega_{X^{[2]}[4]} \simeq j_*(\wedge^4N)[4] \otimes pr_{X^{[2]}[2]}^* \omega_{X^{[2]}[4]} \simeq j_*\mathcal{O}(E);
$$

159
see e.g. [Huy06, Corollary 3.40]. Let us assume that \( \tau \) acts trivially on \( \mathcal{O}_E = \mathcal{O}_E(E) \). Then by [DN89, Theorem 2.3] the sheaf of invariants \( q_\ast \mathcal{O}^\mathbb{G}_2 \) is the descent of \( \mathcal{Q} \), i.e. \( q_\ast \mathcal{Q}^\mathbb{G}_2 \simeq \mathcal{Q} \). We also have \( \Phi_\ast^{-1}(\mathcal{O}_{X^2}) \simeq q_\ast \mathcal{O}^\mathbb{G}_2 \) and by [Sca09a, Proposition 1.3.3] also \( \Phi_\ast^{-1}(\mathcal{O}_{X^2}) \simeq \mathcal{O}_{X^2} \). All this together implies that \( \mathcal{O}(E) \simeq q \ast \mathcal{O}_{X^2} \) is the trivial line bundle which is a contradiction.

Let \( d : X \to S^2X \) be the diagonal embedding and consider the cartesian diagram

\[
\begin{array}{c}
X^2 \xrightarrow{\mu} \mathbb{P}(\Omega_X) \\
\mu \downarrow \downarrow \mu
\end{array}
\]

If the surface \( X \) has trivial canonical bundle, it is known that any line bundle \( L \) on \( X^2 \) is an EZ-spherical object; see [Huy06, Example 8.49 (iv)]. That means that the functor

\[
\tilde{F}_L : D^b(X) \to D^b(X^2) \quad A \mapsto i_\ast (L \otimes \mu_\Delta A)
\]

is spherical. The Fourier–Mukai kernel of \( \tilde{F}_L \) is given by \( \nu_\ast L \in D^b(X \times X^2) \) where

\[
\nu := (\mu_\Delta, i) : X^2 \xrightarrow{\mu_\Delta} X \times X^2 \tag{6.9}
\]

is the closed embedding onto the graph of \( \mu_\Delta \). The map \( \mu_\Delta : X^2 \to X \) equals the \( \mathbb{P}^1 \)-bundle \( \mathbb{P}(\Omega_X) \to X \) and we denote the tautological line bundle by \( \mathcal{O}_{\mu_\Delta}(1) \in \text{Pic} X^2 \).

Consider the functor \( F = \delta_\ast \text{triv} : D^b(X) \to D^b_{\mathbb{G}_2}(X^2) \) from the previous section as well as \( \hat{F} := M_\delta F : D^b(X) \to D^b_{\mathbb{G}_2}(X^2) \) where \( M_\delta = M_{\mathcal{O}_{X^2} \otimes a} \) is the tensor product with the alternating (or sign) representation \( a \), i.e. the non-trivial character of \( \mathbb{G}_2 \). The functor \( \hat{F} \) is again split spherical (i.e. a \( \mathbb{P}^1 \)-functor) by Lemma 6.2.4(ii).

**Proposition 6.4.2.** Let \( X \) be a smooth projective surface (with arbitrary canonical bundle). Then there are isomorphisms of functors \( \Phi_\ast^{-1} F \simeq \hat{F}_{\mathcal{O}_{\mu_\Delta}(-2)[1]} \) and \( \Phi_\ast^{-1} \hat{F} \simeq \hat{F}_{\mathcal{O}_{\mu_\Delta}(-1)} \).

**Proof.** The FM kernel of \( F \) is \( \mathcal{O}_D \in D^b_{\mathbb{G}_2}(X \times X^2) \) where \( D = \Gamma_\delta \subset X \times X^2 \) is the small diagonal. The composition \( \Phi_\ast^{-1} F \) is the FM transform along the convolution product

\[
\mathcal{Q} \ast \mathcal{O}_D = \text{pr}_{X \times X^2} \ast (\text{pr}_{X \times X^2}^\ast \mathcal{O}_D \otimes \text{pr}_{X \times X^2}^\ast \mathcal{Q})^\mathbb{G}_2
\]

where \( \mathcal{Q} = j_\ast \mathcal{O}(E) \in D^b_{\mathbb{G}_2}(X^2 \times X^2) \) by Lemma 6.4.1. Note that the convolution product of equivariant FM kernels is given by taking invariants of the usual convolution product; see [Plo07, Section 1.2]. Consider the commutative diagram with cartesian squares

\[
\begin{array}{c}
E \xrightarrow{\beta} X \times X^2 \xrightarrow{\text{id} \times j} X \times X^2 \times X^2 \\
\downarrow \downarrow \downarrow \\
D \xrightarrow{j \times \Delta_X \times X^2} X \times X^2.
\end{array}
\]
All the horizontal arrows are the appropriate closed embeddings. In particular, $\beta$ is the embedding of $E \subset X^2 \times X^2$ into $X \times X^2 \times X^2$ as the graph of $p_\Delta := p|_E : E \to \Delta_X \simeq X$.

By the projection formula, we have

$$\text{pr}^*_{X \times X^2} \mathcal{O}_D \otimes \text{pr}^*_{X^2 \times X^2} \mathcal{Q} \simeq (\text{id} \times j)_* \left( (\text{id} \times j)^* \text{pr}^*_{X \times X^2} \mathcal{O}_D \otimes \mathcal{O}(X \times E) \right) \simeq (\text{id} \times j)_* \left( (\text{id} \times p)^* \mathcal{O}_D \otimes \mathcal{O}(X \times E) \right).$$

The morphism $\text{id} \times p$ is the blow up of $X \times X^2$ in $X \times \Delta_X$. By [Huy06, Proposition 11.12] we get

$$\mathcal{H}^*(((\text{id} \times p_\Delta)^* \mathcal{O}_D) \simeq \mathcal{O}_{\pi^{-1}(D)}[0] \oplus (\omega_{\pi} \otimes \mathcal{O}_{\pi}(1))|_{\pi^{-1}(D)}[1] \simeq \mathcal{O}_{\mu_\Delta}[0] \oplus \mathcal{O}_{\mu_\Delta}(-1)[1]$$

where the last isomorphism has to be interpreted using the identification of $\pi^{-1}(D)$ with $E$ given by $\beta$. Since $\mathcal{O}(X \times E)|_{\pi^{-1}(D)} \simeq \mathcal{O}_{\mu_\Delta}(-1)$, we get

$$\mathcal{H}^*(\text{pr}^*_{X \times X^2} \mathcal{O}_D \otimes \text{pr}^*_{X^2 \times X^2} \mathcal{Q}) \simeq \beta_* \mathcal{O}_{\mu_\Delta}(-1)[0] \oplus \beta_* \mathcal{O}_{\mu_\Delta}(-2)[1].$$

The projection $q : I^2 X \to X^2$ gives an isomorphism between the $\mathbb{P}^1$-bundles $p_\Delta : E \to X$ and $\mu_\Delta : X^2 \simeq X$. Thus,

$$\mathcal{H}^*(\text{pr}^*_{X \times X^2} \text{pr}^*_{X^2 \times X^2} \mathcal{O}_D \otimes \text{pr}^*_{X^2 \times X^2} \mathcal{Q}) \simeq \nu_* \mathcal{O}_{\mu_\Delta}(-1)[0] \oplus \nu_* \mathcal{O}_{\mu_\Delta}(-2)[1];$$

see (6.9) for the definition of $\nu$. Since $\tau = (1, 2) \in \mathfrak{S}_2$ acts trivially on $\mathcal{O}_D$, the action on the non-derived pull-back $\mathcal{H}^0((\text{id} \times p)^* \mathcal{O}_D) \simeq \mathcal{O}_{\pi^{-1}(D)}$ is trivial too. Thus the action of $\tau$ on the degree zero term of (6.12) is induced by the linearisation of $\mathcal{Q}$ and hence given by $-1$ (see Lemma 6.4.1) which makes the invariants vanish. If $\mathfrak{S}_2$ were to also act alternatingly on the degree $-1$ term of (6.12), we would have $\mathcal{Q} \ast \mathcal{O}_D = 0$. This would contradict $\Phi^{-1}F = \text{FM} \mathcal{Q} \otimes \mathcal{O}_D$ being a $\mathbb{P}^1$-functor. It follows that $\mathfrak{S}_2$-action on the degree $-1$ term of (6.12) is trivial. Thus, $\mathcal{Q} \otimes \mathcal{O}_D \simeq \nu_* \mathcal{O}_{\mu_\Delta}(-2)[1]$ which proves the first assertion. If we replace $F$ by $\tilde{F} = \text{FM} \mathcal{O}_D \otimes \mathcal{Q}$, the $\mathfrak{S}_2$-action on (6.12) changes by the sign $-1$. It follows that $\Phi^{-1} \tilde{F}$ is the FM transform along $\nu_* \mathcal{O}_{\mu_\Delta}(-1)[0]$ which shows the second assertion. \qed

**Remark 6.4.3.** The proposition says in particular that $\tilde{F}_{\mathcal{O}_{\mu_\Delta}(-2)}$ is also a spherical functor in the case that $\omega_X$ is not trivial. One can also prove this directly and for a general line bundle instead of $\mathcal{O}_{\mu_\Delta}(-2)$.

**Remark 6.4.4.** Since $X^2$ equals the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_X) \to X$, its canonical bundle is $\omega_{X^2} = \mu^* \omega_X^2 \otimes \mathcal{O}_{\mu_\Delta}(-2)$. The Hilbert-Chow morphism $\mu$ is a crepant resolution which means $\omega_X^2 \simeq \mu_* \omega_{X^2}^2$; see e.g. [Nie04, Proposition 1.6]. Thus,

$$\omega_{X^2}(X^2|\Delta) \simeq \mu_* \omega_{X^2}^2 \simeq \mu_* \omega_X^2.$$

It follows by adjunction formula that $\mathcal{O}_{X^2}|(X^2|n) = \mathcal{O}_{\mu_\Delta}(-2)$. The line bundle $\mathcal{O}_{X^2}(-X^2)$ has a square root $B := \mathcal{O}(-X^2/2) \in \text{Pic} X^2$; see [BS91, Appendix]. Its restriction to $X^2$ is of the form $B|X^2| = \mathcal{O}_{\mu_\Delta}(1)$. Using this, we can rewrite for a general line bundle $L = \mu^* K \otimes \mathcal{O}_{\mu_\Delta}(i) \in \text{Pic} X^2$ the spherical functor $\tilde{F}_L$ as $\tilde{F}_L = M^{i+2} \mathcal{O}_{\mu_\Delta}(-2)$. The analogue of Lemma 6.2.4(ii) for spherical functors (see [AA13, Proposition 13]) thus yields $t_L = M^{i+2} \mathcal{O}_{\mu_\Delta}(-2) \mathcal{M}_R^{i+2}$ where $t_L$ denotes the spherical twist associated to $\tilde{F}_L$. In particular, the subgroup $\langle M_B, t_L \rangle \subset \text{Aut}(\mathcal{D}^b(X^2))$ is independent of the line bundle $L \in \text{Pic}(X^2)$.
Proposition 6.5.1. The autoequivalence

**Theorem 6.5.2**

Section 2.3] for details.

it induces an auto equivalence of $\mathcal{D}$

**Embedding** $S$

with Fourier-Mukai kernel

Proof. Let $\phi$ be a morphism that is invariant under the induced action of $\mathfrak{m}$. For every $\phi \in \text{Aut}(X)$ we have $\phi(\varphi^n) = (\varphi^n)_\phi$ where $\varphi^n$ is the $\mathfrak{S}_n$-equivariant automorphism of $X^n$ given by $\varphi^n(x_1, \ldots, x_n) = (\varphi(x_1), \ldots, \varphi(x_n))$. Furthermore, $\phi$ acts on $X^n$ by the morphism $\varphi^n$, which is given by $\varphi^n(\xi) = [\varphi(\xi)]$, and on $X$ by the morphism $\varphi^n$. The isospectral Hilbert scheme $\mathcal{H} X \subset X^n \times X^n$ is invariant under the induced action of $\text{Aut}(X)$ on $X^n \times X^n$. Thus, the Bridgeland–King–Reid–Haiman equivalence $\Phi = \text{FM}_{\mathcal{O}_X}$ is $\text{Aut}(X)$-equivariant in the sense that $\Phi \varphi^*_n \simeq \varphi^n \Phi$. Hence, $\alpha(\varphi_n) \in \text{Aut}((\mathcal{D})_\mathfrak{S}_n(X^n))$ corresponds to $\varphi^*_n \in \text{Aut}(\mathcal{D}(\mathcal{D}^b(X^n)))$. For $L \in \text{Pic} X$ we have $\alpha(M_L) = M^\mathfrak{S}_n L$ where $L^\mathfrak{S}_n$ is considered as an $\mathfrak{S}_n$-equivariant line bundle with the natural
Lemma 6.5.4. 1. For every automorphism $\varphi \in \text{Aut}(X)$ we have $\beta \alpha(\varphi) \simeq \alpha(\varphi) \beta$. In the case $n = 2$, also $\sqrt{3} \alpha(\varphi) \simeq \alpha(\varphi) \sqrt{3}$. 2. For every line bundle $L \in \text{Pic}(X)$ we have $\beta \alpha(L) \simeq \alpha(L) \beta$. If $n = 2$, also $\sqrt{3} \alpha(L) \simeq \alpha(L) \sqrt{3}$.

Proof. We have $\alpha(\varphi) F \simeq F \varphi$ and $\alpha(L) F \simeq M_L \otimes F$. The assertions now follow by Lemma 6.2.4 (for $\sqrt{3}$ one has to use the analogous result [AA13, Proposition 13] for spherical twists).

Let $G \subset \text{Aut}(D^b(X[n]))$ be the subgroup generated by $\beta$, shifts, and $\alpha(\text{Aut}_{st}(D^b(X)))$.

Proposition 6.5.5. The map

$$S: \mathbb{Z} \times \mathbb{Z} \times (\text{Aut}(X) \ltimes \text{Pic}(X)) \to \text{Aut}(D^b_{\text{st}}(X^n)), \quad (k, \ell, \Psi) \mapsto \beta^k \ell \alpha(\Psi)$$

defines a group isomorphism onto $G$.

Proof. By the previous lemma, $\beta$ indeed commutes with $\alpha(\Psi)$ for $\Psi \in \text{Aut}_{st}(D^b(X))$. Together with Theorem 6.5.2 and the fact that shifts commute with every derived autoequivalence, this shows that $S$ is indeed a well-defined group homomorphism with image $G$. Now consider $g = \beta^k \ell \alpha(\varphi) \alpha(M_L)$ and assume $g = \text{id}$. For every point $[\xi] \in X^n \setminus X^n_{\Delta}$ we have $g(\mathbb{C}([\xi])) = \mathbb{C}([\varphi(\xi)])(\ell)$ which shows $\ell = 0$ and $\varphi = \text{id}$. Thus, $g = \beta^k M_L \otimes \mathbb{C}$. By Proposition 6.2.1(ii), for $A \in D^b(X)$ we have $g(FA) = F(A \otimes \omega_X^{kn} \otimes L^n)(-k(2n - 2))$. This shows that the degrees in which $g(FA)$ has non-zero cohomology are exactly the degrees in which $FA$ has non-zero cohomology shifted by $-k(2n - 2)$. Thus, for $g = \text{id}$ to hold we need $k = 0$. Finally, $g = M_L \otimes \mathbb{C}$ is trivial only if $L = \mathcal{O}_X$.

Remark 6.5.6. For $n = 2$, the analogous statement with $\beta$ replaced by $\sqrt{3}$ also holds.

Let now $X$ be a K3 surface. In this case, the Fourier–Mukai transform $F_\alpha: D^b(X) \to D^b(X[n])$ with kernel the universal sheaf $\mathcal{I}_\Xi$ is a $\mathbb{P}^{n-1}$-functor with $\mathbb{P}$-cotwist $D = [-2]$; see [Add16]. Here, $\Xi \subset X \times X^n$ is the universal family of length $n$ subschemes. We denote the associated $\mathbb{P}^{n-1}$-twist by $\gamma$ and in case $n = 2$ the spherical twist by $\sqrt{3}$.

Lemma 6.5.7. For every point $[\xi] \in X^n$ the object $\gamma(\mathbb{C}([\xi]))$ is supported on the whole $X^n$.

In the case $n = 2$, the same holds for the object $\sqrt{3}(\mathbb{C}([\xi]))$.

Proof. We set for short $A = \mathbb{C}([\xi])$ and $F = F_\alpha$. We will use the exact triangle of Fourier–Mukai transforms $F \to F' \to F''$ induced by the exact triangle of kernels $\mathcal{P} \to \mathcal{P}' \to \mathcal{P}''$ with $\mathcal{P} = \mathcal{I}_\Xi$, $\mathcal{P}' = \mathcal{O}_{X \times X^n}$, and $\mathcal{P}'' = \mathcal{O}_\Xi$. The right adjoints form the exact triangle $R'' \to R' \to R$ with kernels $Q'' = \mathcal{O}_\Xi[2]$, $Q' = \mathcal{O}_{X \times X^n}[2]$, and $Q = \mathcal{I}_\Xi[2]$. Since $\Xi$ is flat over $X[n]$, we get

$$R''(A) = \mathcal{O}_\xi[2] \simeq \mathcal{O}_\xi[0], \quad R'(A) = H^*(X[n], A) \otimes \mathcal{O}_X[2] = \mathcal{O}_X[2].$$

Setting $H^i = H^i(R(A))$, the long exact cohomology sequence associated to the triangle $R''(A) \to R'(A) \to R(A)$ gives $H^{-2} = \mathcal{O}_X$, $H^{-1} = \mathcal{O}_\xi$, and $H^0 = 0$ for all other values.
of $i$. The only functor in the composition $F = \text{pr}_{X[n]}^{*} (\text{pr}_{X}^{*}(\_ \otimes \mathcal{I}_{\Xi})$ that needs to be derived is the push-forward along $\text{pr}_{X[n]}$. The reason is that the non-derived functors $\text{pr}_{X}^{*}$ as well as $\text{pr}_{X}^{*}(\_ \otimes \mathcal{I}_{\Xi}$ are exact; see [Sca09b, Proposition 2.3] for the latter. Thus, there is the spectral sequence
\[ E_{2}^{pq} = \mathcal{H}^{p}(F(H^{q})) \Longrightarrow E_{p+q}^{pq} = \mathcal{H}^{p+q}(FR(A)) \]
associated to the derived functor $\text{pr}_{X[n]}^{*}$; see e.g. [Huy06, Proposition 2.66]. It is zero outside of the rows $q = -1$ and $q = -2$. Now $F'(\mathcal{O}_{\Xi}) = H^{*}(X, \mathcal{O}_{\Xi}) \otimes \mathcal{O}_{X[n]} = \mathcal{O}_{X[n]}^{|X[n]|[0]}$ and $F''(\mathcal{O}_{\Xi})$ is also concentrated in degree zero since $\Xi$ is finite over $X[n]$. By the long exact sequence associated to $F(H^{-1}) \rightarrow F'(H^{-1}) \rightarrow F''(H^{-1})$ we see that all terms in the $q = -1$ row except for $E_{2}^{0,-1}$ and $E_{2}^{1,-1}$ must vanish. Furthermore,
\[ F'(H^{-2}) = H^{*}(X, \mathcal{O}_{X}) \otimes \mathcal{O}_{X[n]} = \mathcal{O}_{X[n]}[0] \oplus \mathcal{O}_{X[n]}[-2] \]
and $F''(H^{-2})$ is a locally free sheaf of rank $n$ concentrated in degree zero since $\Xi$ is flat of degree $n$ over $X[n]$. This shows that the $-2$ row of $E_{2}$ is zero outside of degrees 0, 1, and 2 and that $E_{2}^{1,-2}$ is of positive rank. The positions of the non-zero $E_{2}$ terms force $E_{2}^{1,-2} = E_{\infty}^{1,-2}$. Thus, $E_{-1} = \mathcal{H}^{-1}(FR(A))$ is of positive rank. Furthermore, we can read off the spectral sequence that the cohomology of $FR(A)$ is concentrated in the degrees $-2$, $-1$, and 0. The $\mathbb{P}$-cotwist of $F$ is given by $D = [-2]$. Thus, the cohomology of $FDR(A)$ is concentrated in the degrees 0, 1, 2. By the long exact sequences associated to the cones occurring in the definition of the spherical and the $\mathbb{P}$-twist (see Section 6.2) it follows that $\mathcal{H}^{-2}(\sqrt{\gamma}(A))$ as well as $\mathcal{H}^{-2}(\gamma(A))$ are of positive rank.

**Proposition 6.5.8.**

1. The subgroup $U \subset \text{Aut}(\mathcal{D}(X[n]))$ generated by the $\mathbb{P}$-twist $\gamma$ and push-forwards along natural automorphisms, i.e. autoequivalences of the form $\varphi_{*}^{[n]} = \alpha(\varphi_{*})$, is isomorphic to $\mathbb{Z} \times \text{Aut}(X)$.

2. $\beta \notin U = \langle \gamma, \{\varphi_{*}^{[n]}\}_{\varphi \in \text{Aut}(X)} \rangle$.

3. $\gamma \notin G = \langle \beta, \{\ell\}_{\ell \in \mathbb{Z}}, \alpha(\text{Aut}_{\text{aff}}(\mathcal{D}(X))) \rangle$.

**Proof.** We have for $\varphi \in \text{Aut}(X)$ that $\varphi_{*}^{[n]} F_{a} = F_{a} \varphi_{*}$ which by Lemma 6.2.4 shows that $\gamma$ commutes with $\varphi_{*}^{[n]}$. The reason is that the subvariety $\Xi \subset X \times X[n]$ is invariant under the morphism $\varphi \times \varphi^{[n]}$. By Proposition 6.2.1(ii) we have $\gamma^{k} \varphi_{*}^{[n]}(F_{a}(A)) \simeq F_{a}(\varphi_{*}A)[-k(2n-2)]$ for $A \in \mathcal{D}(X)$. Thus, $\gamma^{k} \varphi_{*}^{[n]} = \text{id}$ implies $k = 0$ and $\varphi = \text{id}$. This means that there are no further relations in the group $U$ which shows (i). The autoequivalence $g = \gamma^{k} \varphi_{*}^{[n]} \in U$ has $g(F_{a}(\mathcal{O}_{X})) = F_{a}(\mathcal{O}_{X})[-k(2n-2)]$. Thus, by Remark 6.2.7 the equality $\beta = g$ implies $k = 1$. But also $\beta = \gamma \varphi_{*}^{[n]}$ cannot hold. Indeed, for $[\xi] \in X[n] \setminus X_{\Delta}$ we have $\beta([\xi]) = \mathcal{C}([\xi])$ by Remark 6.4.5 and Proposition 6.2.1(i). In contrast, $\gamma \varphi_{*}^{[n]}([\xi])$ is supported on the whole $X[n]$ by Lemma 6.5.7. The assertion (iii) is shown similarly by comparing the values of the autoequivalences $\gamma$ and $g \in G$ on skyscraper sheaves.

**Remark 6.5.9.** The same results hold for $\gamma$ replaced by $\sqrt{\gamma}$ and $\beta$ replaced by $\sqrt{\beta}$ in the case $n = 2$.

Using the same arguments as in [Add16, Sections 1.4 & 3.4] one can also show that $\beta$ does not equal a shift of an autoequivalence induced by a $\mathbb{P}^{n}$-object on $X[n]$ or of an autoequivalence of the form $\alpha(T_{E})$ for a spherical twist $T_{E}$ on the surface. In particular, $\beta$ is an exotic autoequivalence in the sense of [PS14].
6.6 \( \mathbb{P}^n \)-objects on generalised Kummer varieties

Let \( A \) be an abelian surface. There is the summation map

\[
\Sigma: \mathbb{P}^n \rightarrow A, \quad (a_1, \ldots, a_n) \mapsto \sum_{i=1}^{n} a_i.
\]

We set \( N_{n-1}A := \Sigma^{-1}(0) \). It is isomorphic to \( A^{n-1} \) via e.g. the morphism

\[
A^{n-1} \rightarrow N_{n-1}A, \quad (a_1, \ldots, a_{n-1}) \mapsto (a_1, \ldots, a_{n-1}, -\sum_{i=1}^{n-1} a_i).
\]

The subvariety \( N_{n-1} \subset A^n \) is \( \Theta_n \)-invariant. Thus, we have \( N_{n-1}A/\Theta_n \subset S^n A \). The generalised Kummer variety is defined as \( K_{n-1}A := \mu^{-1}(N_{n-1}A/\Theta_n) \), i.e. it is the subvariety of the Hilbert scheme \( A^{[n]} \) consisting of all points representing subschemes whose weighted support adds up to zero. It can be identified with \( \text{Hilb}^S(N_{n-1}A) \) and also the other assumptions of the Bridgeland–King–Reid Theorem are satisfied which leads to the equivalence

\[
\bar{\Phi} = \text{FM}_{\Theta_n^A} : D^b(K_{n-1}A) \rightarrow D^b_{\Theta_n}(N_{n-1}A)
\]

where \( \bar{P}^nA = p^{-1}(N_{n-1}A) \); see [Nam02, Remark 3] or [Mea15, Lemma 6.2]. The intersection between the small diagonal \( \Delta = \delta(A) \subset A^n \) and \( N_{n-1}A \) consists exactly of the points \( \delta(a) = (a, \ldots, a) \) for \( a \) an \( n \)-torsion point of \( A \), i.e. \( \Delta \cap N_{n-1}A = \delta(A_n) \) where \( A_n \subset A \) denotes the set of \( n \)-torsion points. The intersection is transversal since under the identification \( T_{A^n} \cong T_{\Delta}^{\otimes n} \) the tangent space of \( \Delta \) in a point \( \delta(a) \) with \( a \in A_n \) is given by vectors of the form \( (v, \ldots, v) \in T_{A}(a)^{\otimes n} \) whereas the tangent space of \( N_{n-1}A \) is given by vectors \( (v_1, \ldots, v_n) \in T_{A}(a)^{\otimes n} \) with \( \sum_{i=1}^{n} v_i = 0 \). We get for the tangent space of \( N_{n-1}A \) in \( \delta(a) \) the identification \( T_{N_{n-1}A}(\delta(a)) \cong N_{\Delta/A^n}(\delta(a)) \). Since the \( \Theta_n \)-action on \( N_{n-1}A \) is just the restriction of the action on \( A^n \), this isomorphism is equivariant.

**Theorem 6.6.1.** Let \( n \geq 2 \). For every \( n \)-torsion point \( a \in A_n \) the skyscraper sheaf \( \mathcal{C}(\delta(a)) \) is a \( \mathbb{P}^{n-1} \)-object in \( D^b_{\Theta_n}(N_{n-1}A) \).

**Proof.** Indeed, using the results for the invariants of \( \wedge^*N_{\Delta/A^n} \) of Section 6.3

\[
\Hom_{D^b_{\Theta_n}(N_{n-1}A)}(\mathcal{C}(\delta(a)), \mathcal{C}(\delta(a))) \cong \Ext^*_A(\mathcal{C}(\delta(a)), \mathcal{C}(\delta(a))) \cong [\wedge^*T_{N_{n-1}A}(\delta(a))]^{\otimes n} \cong \wedge^*N_{\Delta/A^n}(\delta(a)) \cong \mathbb{C} \oplus \mathbb{C}[-2] \oplus \cdots \oplus \mathbb{C}[-2(n-1)].
\]

The algebra structure has the desired form by Lemma 6.3.3. Since the canonical bundle of \( N_{n-1}A \) is trivial, \( \mathcal{C}(\delta(a)) \) is indeed a \( \mathbb{P}^{n-1} \)-object. \( \square \)

**Remark 6.6.2.** For two different \( n \)-torsion points the skyscraper sheaves are orthogonal which makes the associated twists commute. By Corollary 6.2.5, we have an inclusion \( \mathbb{Z}^{n^2} \subset \text{Aut}(D^b_{\Theta_n}(N_{n-1}A)) \cong \text{Aut}(D^b(K_{n-1}A)) \).
In the case \( n = 2 \) the generalised Kummer variety \( K_{n-1}A = K_1A \) is just the Kummer surface \( K(A) \). Moreover, there is an isomorphism of commutative diagrams

\[
\begin{array}{ccc}
\overline{P^2}A & \xrightarrow{p} & N_1A \\
\Downarrow q & & \Downarrow \pi \\
K_1A & \xrightarrow{\mu} & N_1A/\mathbb{G}_2
\end{array}
\begin{array}{ccc}
\tilde{A} & \xrightarrow{p} & A \\
\Downarrow q & & \Downarrow \pi \\
K(A) & \xrightarrow{\mu} & A/\iota
\end{array}
\]

where \( p \) and \( \mu \) in the right-hand diagram are the blow-ups of the 16 different 2-torsion points and of their image under the quotient under the involution \( \iota = (-1) \), respectively. For a 2-torsion point \( a \in A_2 \) we denote by \( E(a) \) the exceptional divisor over the point \([a] \in A/\iota \). Since \( E(a) \) is a rational curve in the \( K3 \)-surface \( K(A) \), every line bundle on it is a spherical object in \( \mathbb{D}^b(K(A)) \); see [ST01, Example 3.5].

**Proposition 6.6.3.** We have \( \Phi^{-1}(\mathbb{C}(\delta(a))) \simeq \mathcal{O}_{E(a)}(-2)[1] \) and \( \Phi^{-1}(\mathbb{C}(\delta(a))\otimes a) \simeq \mathcal{O}_{E(a)}(-1) \) for every 2-torsion point \( a \in A_2 \).

**Proof.** We have \( i_*\Phi^{-1} \simeq \Phi^{-1}j_* \) for the closed embeddings \( i: K_1A \to A^{[2]} \) and \( j: N_1A \to A^2 \); see [Mea15, Lemma 6.2]. Using Proposition 6.4.2 we get

\[
\Phi^{-1}(\mathbb{C}(\delta(a))) \simeq \Phi^{-1}(j_*\mathbb{C}(\delta(a))) \simeq \Phi^{-1}(F(\mathbb{C}(a))) \simeq \Phi_{i}\mathcal{O}_{\Delta(-2)}(\mathbb{C}(a))[1] \simeq i_*\mathcal{O}_{E(a)}(-2)[1].
\]

Since the push-forward along a closed embedding is exact, we have \( \mathcal{H}^*(i_*B) \simeq i_*\mathcal{H}^*(B) \) for every object \( B \in \mathbb{D}^b(K_1A) \). Thus, \( i_*\Phi^{-1}(\mathbb{C}(\delta(a))) \simeq i_*\mathcal{O}_{E(a)}(-2)[1] \) implies \( \Phi^{-1}(\mathbb{C}(\delta(a))) \simeq \mathcal{O}_{E(a)}(-2)[1] \). The proof of the second assertion is the same. \( \square \)

There is no known homomorphism \( \text{Aut}(\mathbb{D}^b(A)) \to \text{Aut}(\mathbb{D}^b(K_nA)) \) analogous to Ploog’s map \( a \). But at least one can lift line bundles \( L \in \text{Pic} A \) (by restricting \( \mathcal{D}_L \) and group automorphisms \( \varphi \in \text{Aut}(A) \) (by restricting \( \varphi^{[n]} \)) to the generalised Kummer variety. Recently, Meachan has shown in [Mea15] that the restriction of Addington’s functor to the generalised Kummer variety \( K_nA \) for \( n \geq 2 \) (i.e. the Fourier–Mukai transform with kernel the universal sheaf) is still a \( \mathbb{P}^{n-1} \) functor and thus yields an autoequivalence \( \tilde{a} \). Comparing these autoequivalences with those induced by the above \( \mathbb{P}^n \)-objects one gets results similar to the results of Section 6.5.

**References**


Chapter 7

\( \mathbb{P} \)-functor versions of the Nakajima operators

(arXiv:1405.1006.)

Abstract

For a smooth quasi-projective surface \( X \) we construct a series of \( \mathbb{P}^{n-1} \)-functors \( H_{\ell,n} : D^b(X \times X^{[\ell]}) \to D^b(X^{[n+\ell]}) \) for \( n > \ell \) and \( n > 1 \) using the derived McKay correspondence. They can be considered as analogues of the Nakajima operators. The functors also restrict to \( \mathbb{P}^{n-1} \)-functors on the generalised Kummer varieties. We also study the induced autoequivalences and obtain, for example, a universal braid relation in the groups of derived autoequivalences of Hilbert squares of K3 surfaces and Kummer fourfolds.

7.1 Introduction

A central result in the theory of Hilbert schemes of points on surfaces is the identification of their cohomology with the Fock space representation of the Heisenberg algebra by means of the Nakajima operators \( q_{\ell,n} : H^*(X \times X^{[\ell]}, \mathbb{Q}) \to H^*(X^{[n+\ell]}, \mathbb{Q}) \); see [Nak97] and [Gro96].

There has been effort towards lifting this action from cohomology to other invariants of the Hilbert schemes, in particular to \( K \)-theory and the derived category. There are results when \( X \) is the affine plane [FT11], [SV13] or a minimal resolution of a Kleinian singularity [CL12].

More recently, autoequivalences of the (bounded) derived categories \( D^b(X^{[n]}) \) of the Hilbert schemes were intensively studied; see [Plo07], [Add16], [PS14], [Mea15], [6], [CLS14], [5]. In particular, the notion of \( \mathbb{P}^n \)-functors was introduced in [Add16]. These are Fourier–Mukai transforms \( F : D^b(M) \to D^b(N) \) between derived categories of varieties having a right-adjoint \( F^R : D^b(N) \to D^b(M) \) and the main property that

\[
F^R \circ F \cong \text{id} \oplus D \oplus D^2 \oplus \cdots \oplus D^n
\]

for some autoequivalence \( D : D^b(M) \to D^b(M) \) which is then called the \( \mathbb{P} \)-cotwist of \( F \). Every \( \mathbb{P}^n \)-functor \( F \) gives an induced \( \mathbb{P} \)-twist \( P_F : D^b(N) \to D^b(N) \) which is an autoequivalence.
The main example given in [Add16] is the $\mathbb{P}^{n-1}$-functor $F_n = \text{FM}_{\mathcal{I}_\Xi} : \mathbb{D}^b(X) \to \mathbb{D}^b(X^{[n]})$ which is given by the Fourier–Mukai transform along the ideal sheaf of the universal family $\Xi \subset X \times X^{[n]}$ for $X$ a K3 surface and $n \geq 2$. Similarly, let $A$ an abelian surface and let $\Xi \subset A \times K_nA$ denote the universal family of the generalised Kummer variety $K_nA \subset A^{[n+1]}$. Then the functor $\tilde{F}_{n+1} = \text{FM}_{\mathcal{I}_\Xi} : \mathbb{D}^b(A) \to \mathbb{D}^b(K_nA)$ is a $\mathbb{P}^{n-1}$-functor for $n \geq 2$ and a $\mathbb{P}^1$-functor for $n = 1$; see [Mea15] and [3]. We will refer to these examples of $\mathbb{P}$-functors as the universal ideal functors.

An important tool for the investigation of the derived categories of points on surfaces is the Bridgeland–King–Reid–Haiman equivalence (also known as derived McKay correspondence; see [BKR01] and [Hai01])

$$\Phi = \Phi_n : \mathbb{D}^b(X^{[n]}) \xrightarrow{\sim} \mathbb{D}^b_{\mathcal{G}_n}(X^n).$$

This provides an identification of $\mathbb{D}^b(X^{[n]})$ with the derived category of equivariant coherent sheaves on the product $X^n$, i.e. of coherent sheaves equipped with a linearisation by the symmetric group $\mathcal{G}_n$.

In [6] it was shown that the composition $H_{0,n} : \mathbb{D}^b(X) \xrightarrow{\text{triv}} \mathbb{D}^b_{\mathcal{G}_n}(X) \xrightarrow{\delta} \mathbb{D}^b_{\mathcal{G}_n}(X^n)$ is a $\mathbb{P}^{n-1}$-functor for every smooth surface $X$ and $n \geq 2$. Here, the first functor of the composition equips every object with the trivial $\mathcal{G}_n$-linearisation and the second is the push-forward along the embedding of the small diagonal. The $\mathbb{P}$-cotwist of $H_{0,n}$ is $S_X^{-1} := (_) \otimes \omega^{-1}_X[-2]$. Under the BKRH equivalence, this functor corresponds to the $\mathbb{P}^{n-1}$-functor $\Phi^{-1} \circ H_{0,n} : \mathbb{D}^b(X) \to \mathbb{D}^b(X^{[n]})$ whose FM-kernel is supported on $Z^{0,n} = \{(x, [\xi]) \mid \text{supp}(\xi') = \{x\}\}$. Thus, one can regard $H_{0,n}$ as a lift of the Nakajima operator $q_{0,n} : \mathbb{H}^n(X, \mathbb{Q}) \to \mathbb{H}^n(X^{[n]}, \mathbb{Q})$.

### 7.1.1 Main results

The question is whether the other Nakajima operators also have analogues in the form of $\mathbb{P}^{n-1}$-functors

$$H_{\ell,n} : \mathbb{D}^b_{\mathcal{G}_n}(X \times X^{\ell}) \cong \mathbb{D}^b(X \times X^{[\ell]}) \to \mathbb{D}^b(X^{[n+\ell]}) \cong \mathbb{D}^b_{\mathcal{G}_{n+\ell}}(X^{n+\ell}).$$

A first natural guess for a generalisation of $H_{0,n}$ would be the functor

$$H_{\ell,n}^0 : \mathbb{D}^b_{\mathcal{G}_n}(X \times X^{\ell}) \xrightarrow{\text{triv}} \mathbb{D}^b_{\mathcal{G}_n \times \mathcal{G}_\ell}(X \times X^{\ell}) \xrightarrow{\delta_{[\ell]*}} \mathbb{D}^b_{\mathcal{G}_n \times \mathcal{G}_\ell}(X^{n} \times X^{\ell}) \xrightarrow{\text{Inf}} \mathbb{D}^b_{\mathcal{G}_{n+\ell}}(X^{n+\ell}); \quad (7.2)$$

see Section 7.2.1 for details on the inflation functor $\text{Inf}$ and its adjoint $\text{Res}$. Here, for a subset $J \subset [n + \ell] = \{1, \ldots, n + \ell\}$ of cardinality $|J| = n$ the morphism $\delta_J$ denotes the closed embedding of the partial diagonal

$$X \times X^{\ell} \cong D \Delta_J = \{(y_1, \ldots, y_{n+\ell}) \mid y_a = y_b \text{ for all } a, b \in J\} \subset X^{n+\ell}.$$

In fact, for every bijection $\mu : [\ell] \to \tilde{J} := [n+\ell] \setminus J$ there is the embedding $\delta_{J,\mu} : X \times X^{\ell} \to X^{n+\ell}$ onto $\Delta_J$ given by

$$\delta_{J,\mu}(x, x_1, \ldots, x_{\ell}) = (y_1, y_{\mu(1)}, \ldots, y_{\mu(\ell)}, \ldots, y_\ell) \quad y_j = x \forall j \in J, \ y_{\mu(i)} = x_i \forall i \in [\ell]$$

and we set $\delta_J = \delta_{J,e}$ where $e : [\ell] \to \tilde{J}$ denotes the unique strictly increasing bijection. The functor $H_{\ell,n}^0$ is given on objects by

$$H_{\ell,n}^0(E) = \bigoplus_{J \subset [n+\ell], |J| = n} \delta_{J,e}(E) \quad \text{for } E \in \mathbb{D}^b_{\mathcal{G}_n}(X \times X^{\ell}). \quad (7.3)$$
The first part of the composition (7.2), namely \( \delta[n] \circ \text{triv} \), can be rewritten as \( H_{0,n} \otimes \text{id}_{X^\ell} \). Since \( H_{0,n} \) is a \( \mathbb{P}^{n-1} \)-functor so is \( \delta[n] \circ \text{triv} \) and
\[
(\delta[n] \circ \text{triv})^R(\delta[n] \circ \text{triv}) \cong S_X^{-[0,n-1]} := \text{id} \oplus S_X^{-1} \oplus \cdots \oplus S_X^{-(n-1)}
\]
where we write \( S_X := (\_ \otimes (\omega_X \otimes \mathcal{O}_X))[2] \) even in the case that \( X \) is not projective. But for the computation of \( H_{0,n}^R \circ H_{\ell,n} \) also the other summands of (7.3) besides \( \delta[n]E \) have to be taken into account which yields
\[
H_{0,n}^R H_{\ell,n}^0(E) \cong S_X^{-[0,n-1]}(E) \oplus (\text{terms supported on partial diagonals of } X \times X^\ell).
\] (7.4)
The approach is to adapt \( H_{\ell,n}^0 \) slightly in order to erase the error term in (7.4). We succeed in doing so by replacing \( H_{\ell,n}^0 \) by a complex of functors
\[
H_{\ell,n} := (0 \to H_{\ell,n}^0 \to H_{\ell,n}^1 \to \cdots \to H_{\ell,n}^\ell \to 0).
\]
More concretely, this means the following. First, we have to set \( H_{\ell,n}^0 := \text{Inf} \circ \delta[n] \circ \text{M}_{a_n} \circ \text{triv} \).
That means that the definition of \( H_{\ell,n}^0 \) differs from (7.2) by
\[
\mathcal{M}_{a_n} := (\_ \otimes C \mathfrak{a}_n : \mathcal{D}_0^b \otimes \mathfrak{S}_\ell \times \mathfrak{S}_{n+\ell}(X \times X^\ell) \to \mathcal{D}_0^b \otimes \mathfrak{S}_\ell \times \mathfrak{S}_{n+\ell}(X \times X^\ell)
\]
where the alternating representation \( a_n \) is the non-trivial character of \( \mathfrak{S}_n \). The FM kernel of \( H_{\ell,n}^0 \) is given by
\[
\mathcal{P}_{\ell,n} := \bigoplus_{J \subseteq [n+\ell], |J| = n} \mathcal{O}_{\Gamma_{\delta,J,\mu}} \otimes \mathfrak{a}_J \in \mathcal{D}_\ell^b \times \mathfrak{S}_{n+\ell}(X \times X^\ell \times X^{n+\ell}).
\]
In Section 7.2.4 we construct a complex \( \mathcal{P}_{\ell,n} = (0 \to \mathcal{P}_{\ell,n}^0 \to \cdots \to \mathcal{P}_{\ell,n}^\ell \to 0) \) whose terms \( \mathcal{P}_{\ell,n}^i \) for \( 1 \leq i \leq \ell \) are direct sums of structure sheaves of certain subvarieties of the graphs \( \Gamma_{\delta,J,\mu} \).
Then we set \( H_{\ell,n} := \text{FM}_{\mathcal{P}_{\ell,n}} \). The definition of the complex \( \mathcal{P}_{\ell,n} \in \mathcal{D}_b^b \times \mathfrak{S}_{n+\ell}(X \times X^\ell \times X^{n+\ell}) \) makes sense for \( X \) a variety of arbitrary dimension and our first result is

**Proposition A.** Let \( X = C \) be a smooth curve and \( n > \max\{\ell, 1\} \).

1. We have \( H_{\ell,n}^R \circ H_{\ell,n} \cong \text{id} \) which means that \( H_{\ell,n} \) is fully faithful.

2. Let \( \ell', n' \) be positive integers with \( n' + \ell' = n + \ell \) and \( \ell' > \ell \). Then \( H_{\ell',n}^R \circ H_{\ell,n} = 0 \).

**Corollary B.** Let \( m \geq 2 \). For \( m \) even we set \( r = \frac{m}{2} - 1 \) and for \( m \) odd we set \( r = \frac{m-1}{2} \). For every smooth curve \( C \) there is a semi-orthogonal decomposition
\[
\mathcal{D}_b^b \mathfrak{S}_m(C^m) = \langle \mathcal{A}_{0,m}, \mathcal{A}_{1,m-1}, \ldots, \mathcal{A}_{r,m-r}, B \rangle
\]
where \( \mathcal{A}_{\ell,m-\ell} := H_{\ell,m-\ell}(\mathcal{D}_b^b(C \times C^\ell)) \cong \mathcal{D}_b^b(C \times C^\ell) \).

We will see in Section 7.5.8 that the fully faithful functors \( H_{\ell,m-\ell} \) also induce autoequivalences of \( \mathcal{D}_b^b \mathfrak{S}_m(C^m) \). Note that for \( C = E \) an elliptic curve the canonical bundle of the product \( E^m \) is trivial. But as a \( \mathfrak{S}_m \)-bundle it is given by \( \mathcal{O}_{E^m} \otimes a_n \) which means that the canonical bundle of the quotient stack \( [E^m/\mathfrak{S}_m] \) is non-trivial. Otherwise, \( \mathcal{D}_b^b([E^m/\mathfrak{S}_m]) \cong \mathcal{D}_b^b(E^m) \) could not allow a semi-orthogonal decomposition.

To us, the case of most interest is that of a smooth quasi-projective surface due to the BKRH equivalence \( \mathcal{D}_b^b(X^{[m]}) \cong \mathcal{D}_b^b(X^m) \).
Theorem C. Let $X$ be a smooth surface and $n > \max\{\ell, 1\}$. Then $H_{\ell,n}$ is a $\mathbb{P}^{n-1}$-functor with $\mathbb{P}$-cotwist $\tilde{S}_X$. In particular, $H_{\ell,n}^R \circ H_{\ell,n} \cong \tilde{S}_X^{[0,n-1]}$.

In the case that $X = A$ is an abelian variety, there is also the following variant of the above construction. For $m \geq 2$ we consider the $\mathcal{G}_m$-invariant subvariety

$$A^{m-1} \cong N_{m-1}A := \{(a_1, \ldots, a_m) \mid a_1 + \cdots + a_m = 0\} \subset A^m.$$ 

If $A$ is an abelian surface there is a variant of the Bridgeland–King–Reid–Haiman equivalence as an equivalence $\mathbb{D}^b(K_{m-1}A) \cong \mathbb{D}_{\mathcal{G}_m}(N_{m-1}A)$ where $K_{m-1}A \subset A^{[m]}$ is the generalised Kummer variety; see [Nam02] or [Mea15]. We also consider the $\mathcal{G}_2$-invariant subvariety

$$M_{\ell,n} := \{(a, a_1, \ldots, a_\ell) \mid n \cdot a + a_1 + \cdots + a_\ell = 0\} \subset A \times A^\ell.$$ 

Then for all $n \geq 2$ the functor $H_{\ell,n} : \mathbb{D}_{\mathcal{G}_m}(A \times A^\ell) \to \mathbb{D}_{\mathcal{G}_m}(A^{n+\ell})$ restricts to a functor $\hat{H}_{\ell,n} : \mathbb{D}_{\mathcal{G}_m}(M_{\ell,n}A) \to \mathbb{D}_{\mathcal{G}_m}(N_{n+1}A)$; see Section 7.4.12 for details.

For $\ell = 0$ we have $\tilde{M}_{0,n}A = A_n \subset A$ where $A_n$ denotes the set of $n$-torsion points. The functor $\hat{H}_{0,n}$ is given by sending the skyscraper sheaf $\mathcal{C}(a)$ of $a \in A_n$ to the object $\mathcal{C}(a, \ldots, a) \otimes a_n \in \mathbb{D}_{\mathcal{G}_1}(N_{n-1}A)$. As shown in [6], for $A$ an abelian surface the objects $\mathcal{C}(a, \ldots, a) \otimes a_n$ are $\mathbb{P}^{n-1}$-objects in the sense of [HT06]. Similarly, for $A = E$ an elliptic curve they are exceptional.

In contrast, for $\ell \geq 1$ we have a (not $\mathcal{G}_2$-equivariant) isomorphism $M_{\ell,n}A \cong A^\ell$ so that the domain category of the functor $\hat{H}_{\ell,n}$ is indecomposable.

Proposition A'. Let $A = E$ be an elliptic curve and $n > \max\{\ell, 1\}$.

1. We have $\hat{H}_{\ell,n}^R \circ \hat{H}_{\ell,n} \cong \id$ which means that $\hat{H}_{\ell,n}$ is fully faithful.

2. Let $\ell', n'$ be positive integers with $n' + \ell' = n + \ell$ and $\ell' > \ell$. Then $\hat{H}_{\ell',n'}^R \circ \hat{H}_{\ell,n} = 0$.

Corollary B'. For every elliptic curve $E$ there is a semi-orthogonal decomposition

$$\mathbb{D}_{\mathcal{G}_m}(N_{m-1}E) = \langle \mathcal{C}(a, \ldots, a) \otimes a_n \mid a \in A_m, \hat{A}_{1,m-1}, \ldots, \hat{A}_{r,m-1}, \tilde{B} \rangle$$

where $\hat{A}_{\ell,m-\ell} := \hat{H}_{\ell,m-\ell}(\mathbb{D}_{\mathcal{G}_\ell}(M_{\ell,m-\ell}A)) \cong \mathbb{D}_{\mathcal{G}_\ell}(M_{\ell,m-\ell}A)$.

Theorem C'. Let $A$ be an abelian surface and $n > \max\{\ell, 1\}$. Then $\hat{H}_{\ell,n}$ is a $\mathbb{P}^{n-1}$-functor with $\mathbb{P}$-cotwist $[-2]$. In particular, $H_{\ell,n}^R \circ H_{\ell,n} \cong \id \oplus [-2] \oplus \cdots \oplus [-2(n-1)]$.

7.1.2 Structure of the proof

The approach is to compute the compositions $H_{\ell,n}^R \circ H_{\ell,n}$ in order to deduce formulae for $H_{\ell,n}^R \circ H_{\ell,n}$ and finally for $H_{\ell,n}^R \circ H_{\ell,n}$. The general proof may appear complicated due to the sheer number of direct summands occurring. We try to provide intuition by performing some calculations of the functor compositions for $\ell = 1, 2$ in Sections 7.3.6 – 7.3.9. However, we will not determine the induced maps between the $H_{\ell,n}^R H_{\ell,n}^j$ in these subsections so that the arguments for the final computations of $H_{\ell,n}^R H_{\ell,n}$ will be a bit vague as explained in Section 7.3.4.

In Section 7.4 we give rigorous proofs for general $\ell$ and $n$ by doing the computations on the level of the FM kernels.
For $m \geq 2$ let $p_m$ be the standard representation of the symmetric group $S_m$. One can say that the reason for the different behaviour of $H_{t,n}$ in the curve and in the surface case lies in the difference of the $S_m$-representations $\wedge^i p_m$ and $\wedge^i (p_m^\otimes 2)$; compare Lemma 7.3.8. The former is always an irreducible representation while the latter has a one-dimensional subspace of invariants for $0 \leq i \leq 2(m - 1)$ even. For $d \geq 3$ the subspace of invariants of $\wedge^i (p_m^\otimes d)$ is in general higher-dimensional which explains that the shape of $H_{t,n} \circ H_{t,n}$ is not that nice if $\dim X \geq 3$; see Remark 7.3.9.

7.1.3 Similarities to the Nakajima operators

Let $X$ be a smooth quasi-projective surface. In order to justify the title of our paper, we will explain some similarities between the $\mathbb{P}^{n-1}$-functors $H_{t,n}$ and the Nakajima operators $q_{t,n}$.

As indicated above, the $\mathbb{P}^{n-1}$-functors $H_{t,n}$ correspond under the BKR equivalence to $\mathbb{P}^{n-1}$ functors $\Phi_{m+\ell}^{-1} \circ H_{t,n} \circ (\text{id} \otimes \Phi_\ell): D^b(X \times X[\ell]) \to D^b(X[n+\ell])$ which we will often again denote by $H_{t,n}$. The FM kernel of the BKR equivalence $\Phi_m$ is the structure sheaf $O_{\text{iso}} X$ of the isospectral Hilbert scheme

$$I^m X = \{ (\xi, x_1, \ldots, x_m) \mid \mu(\xi) = x_1 + \cdots + x_m \} \subset X[m] \times X$$

where $\mu: X[n] \to S^n X = X^m / S_m$ denotes the Hilbert–Chow morphism and points in the symmetric product are written as formal sums. One can deduce that the Fourier–Mukai kernel of $\Phi_{m+\ell}^{-1} \circ H_{t,n} \circ (\text{id} \otimes \Phi_\ell)$ is supported on $Z_{t,n}$, the correspondence defining the Nakajima operator $q_{t,n}$. Clearly, it would be desirable to have a more concrete description of the FM kernel as an object in $D^b(X \times X[\ell] \times X[n+\ell])$ but for the time being we are unable to provide one except for the case that $\ell = 0$ and $n = 2$; see [6, Section 4].

In the case that $X$ is projective with trivial canonical bundle $\omega_X \cong O_X$, the functors $H_{t,n}$ also fulfill some of the Heisenberg relations on the level of the Grothendieck groups. For $\alpha \in H^*(X, \mathbb{Q})$ there is by the Küneth formula the map

$$i_{\alpha}: H^*(X[\ell], \mathbb{Q}) \to H^*(X \times X[\ell], \mathbb{Q}) \cong H^*(X, \mathbb{Q}) \otimes H^*(X[\ell], \mathbb{Q}) , \quad i_{\alpha}(\beta) = \alpha \otimes \beta .$$

The operators $q_{t,n}(\alpha) := q_{t,n} \circ i_{\alpha}: H^*(X[\ell], \mathbb{Q}) \to H^*(X[n+\ell], \mathbb{Q})$ are again called Nakajima operators. Furthermore, $q_{t,-n}(\alpha): H^*(X[n+\ell], \mathbb{Q}) \to H^*(X[\ell], \mathbb{Q})$ is defined as the adjoint of $q_{t,n}(\alpha)$ with respect to the intersection pairing. One usually considers all of these operators for varying values of $\ell$ together as operators on $H := \oplus_{\ell \geq 0} H^*(X[\ell], \mathbb{Q})$ by setting

$$q_\ell(\alpha) := \oplus_{\ell} q_{t,n}(\alpha): \oplus_{\ell} H^*(X[\ell], \mathbb{Q}) \to \oplus_{\ell} H^*(X[n+\ell], \mathbb{Q}) ,$$

$$q_{-\ell}(\alpha) := \oplus_{\ell} q_{t,-n}(\alpha): \oplus_{\ell} H^*(X[n+\ell], \mathbb{Q}) \to \oplus_{\ell} H^*(X[\ell], \mathbb{Q}) .$$

Then, as shown in [Nak97], the commutator relations between these operators are given by

$$[q_\ell(\alpha), q_{\ell'}(\beta)] = n \cdot \delta_{n,-n'}(\alpha, \beta) \cdot \text{id}_H . \quad (7.5)$$

Taking $n = -n'$ and considering the degree $\ell$ piece of formula (7.5) for $\ell < n$ we get

$$q_{\ell,-n}(\alpha) \circ q_{t,n}(\beta) = n \cdot (\alpha, \beta) \cdot \text{id}: H^*(X[\ell], \mathbb{Q}) \to H^*(X[\ell], \mathbb{Q}) . \quad (7.6)$$

Let $X \xleftarrow{\phi} X \times X[\ell] \xrightarrow{\rho} X[\ell]$ be the projections. For $E \in D^b(X)$ we consider the functor

$$I_E := q^* E \otimes p^*(\quad): D^b(X[\ell]) \to D^b(X \times X[\ell]) , \quad I_E(F) = E \boxtimes F .$$
Its right adjoint is \( I_E^R = p_*(q^*E^\vee \otimes (\_)) \). We set

\[
H_{\ell,n}(E) := H_{\ell,n} \circ I_E : \mathcal{D}^b(X^{[\ell]}) \to \mathcal{D}^b(X^{[n+\ell]}).
\]

For \( E, F \in \mathcal{D}^b(X) \) we get by Theorem C in the case that \( \omega_X \) is trivial

\[
H_{\ell,n}(E)^R \circ H_{\ell,n}(F) \cong I_E^R \circ H_{\ell,n} \circ I_F \cong I_E^R \circ I_F([0] \oplus [-2] \oplus \cdots \oplus [-2(n-1)])
\cong (\_ \otimes_{\mathbb{C}} \text{Ext}^*(E, F)([0] \oplus [-2] \oplus \cdots \oplus [-2(n-1)]).
\]

On the level of the Grothendieck group this gives

\[
H_{\ell,n}(E)^R \circ H_{\ell,n}(F) = n \cdot \chi(E, F) \cdot \text{id} : K(X^{[\ell]}) \to K(X^{[\ell]})
\] (7.7)

which fits nicely with (7.6). However, as we will see in Section 7.3.8, not all the compositions \( H_{\ell,n}(E)^R \circ H_{\ell,n}(F) \) with \( n \neq n' \) fit that well with (7.5). Thus, it seems like the collection of the \( H_{\ell,n} \) does not give rise to a categorified action of the Heisenberg algebra.

In [CL12] such a categorified action was constructed in the case that \( X \) is a minimal resolution of a Kleinian singularity. In Section 7.5.2 we will compare the construction of [CL12] to ours.

### 7.1.4 The induced autoequivalences

In the Sections 7.5.3 – 7.5.6 we study the autoequivalences of the derived categories of Hilbert schemes of points on surfaces and generalised Kummer varieties induced by the \( \mathbb{P} \)-functors of the Theorems C and C’, to which we will refer as the Nakajima \( \mathbb{P} \)-functors. We will see that the twists are rather independent from each other and from the subgroup of standard autoequivalences; see Proposition 7.5.4 and (7.54). For \( X \) a K3 surface, the universal ideal \( \mathbb{P} \)-functor can be truncated in a certain way to give another \( \mathbb{P} \)-functor \( \mathcal{D}^b(X) \to \mathcal{D}^b(X^{[n]}) \); see [5, Section 5]. The existence of these truncated universal ideal functors is in some sense explained by the Nakajima functors; see Section 7.5.5. In Section 7.5.6 we show the existence of a universal braid relation in the groups of derived autoequivalences of Hilbert squares of K3 surfaces and Kummer fourfolds. In the final Section 7.5.9 we make a conjecture about certain cases in which we expect the twists along the Nakajima \( \mathbb{P} \)-functors to generate the full group of derived autoequivalences and give an idea which kind of autoequivalences might still wait to be constructed.

**Convention.** We will work over the complex numbers throughout though many parts remain true over more general ground fields.

**Acknowledgements.** The author thanks Nicolas Addington, Will Donovan, Daniel Huybrechts, Ciaran Meachan, David Ploog, and Pawel Sosna for helpful comments and discussions. This work was supported by the SFB/TR 45 of the DFG (German Research Foundation) and in its final stages by the research grant KR 4541/1-1 of the DFG.
Contents

7.2 Definition of the functors

7.2.1 Equivariant Fourier–Mukai transforms

For details on equivariant derived categories and Fourier–Mukai transforms we refer to [BKR01, Section 4] and [Plo07]. Let $G$ be a finite group acting on a variety $M$. Then we denote by $D^b_G(M) := D^b(\text{Coh}_G(M))$ the bounded derived category of the category $\text{Coh}_G(M)$ of coherent $G$-equivariant sheaves. Let $U \subset G$ be a subgroup. Then there is the forgetful or restriction functor $\text{Res}_U^G: D^b_G(M) \to D^b_U(M)$. It has the inflation functor $\text{Inf}_U^G: D^b_U(M) \to D^b_G(M)$ as a left and right adjoint. For $E \in D^b_U(M)$ we have $\text{Inf}_U^G(E) = \oplus_{g \in U \setminus G} g^*E$ with the $G$-linearisation given as a combination of the $U$-linearisation of $E$ and the permutation of the direct summands. In the following we will often simply write $\text{Res}$ and $\text{Inf}$ for these functors when the groups $G$ and $U$ should be clear from the context. In the case that $G$ acts trivially on $M$ there is also the functor $\text{triv}: D^b(M) \to D^b_G(M)$ which equips every object with the trivial $G$-linearisation. Its left and right adjoint is given by the functor of invariants $(_\cdot)^G: D^b_G(M) \to D^b(M)$.

Let $G'$ be a second finite group acting on $M'$. Then every object $P \in D^b_{G' \times G'}(M \times M')$ induces the equivariant Fourier–Mukai transform

$$\text{FM}_P := [\text{pr}_{M'}^* (\text{pr}_M^* (\_ \otimes P))]^{G \times 1}: D^b_G(M) \to D^b_{G'}(M').$$

For example, if $M = M'$ and $G$ acts trivially, the functor $\text{triv}: D^b(M) \to D^b_G(M)$ is the FM transform along $O_\Delta \in D^b_{G \times G}(M \times M)$ and $(_\cdot)^G: D^b_G(M) \to D^b(M)$ is the FM transform along $O_\Delta \in D^b_{G \times 1}(M \times M)$. Note that, for an arbitrary action of $G$ on $M$, the sheaf $O_\Delta$ has a canonical $G_\Delta$-linearisation where $G_\Delta = \{(g, g)\} \subset G \times G$. The identity functor $\text{id}: D^b_G(M) \to D^b_G(M)$ is the Fourier–Mukai transform along the kernel $\text{Inf}_{G \times G}^G O_\Delta \cong \oplus_{g \in G} O_{\Gamma_g}$, where $\Gamma_g$ denotes the graph of the action $g: X \to X$. More generally, for $E \in D^b_G(M)$ the tensor product functor $(_\cdot) \otimes E: D^b(M) \to D^b(M)$ is given by $\text{Inf}_{G \times G}^G \delta_* E \cong \oplus_{g \in G} (1 \times g)_* E$ where $\delta = (1 \times 1): M \to M \times M$ is the diagonal embedding. The restriction $\text{Res}_U^G$ and the inflation $\text{Inf}_U^G$ are also FM transforms with kernels $\text{Inf}_{U \times G}^G O_\Delta$ and $\text{Inf}_{U \times G}^G O_\Delta$, respectively.

Let $G''$ be a third finite group acting on $M''$ and $Q \in D^b_{G' \times G''}(M' \times M'')$. Then we have $\text{FM}_Q \circ \text{FM}_P = \text{FM}_{Q \times P}$ where $Q \times P$ is the equivariant convolution product

$$Q \times P = [\text{pr}_{M \times M''}^* (\text{pr}_{M' \times M''}^* Q \otimes \text{pr}_{M \times M'}^* P)]^{1 \times G'' \times 1} \in D^b_{G \times G''}(M \times M'').$$ (7.8)
7.2.2 \( \mathbb{P} \)-functors

Let \( G \) and \( H \) be finite groups acting on varieties \( M \) and \( N \). Following \[Add16\], a \( \mathbb{P}^n \)-functor is an (equivariant) Fourier–Mukai transform \( F: \mathbb{D}^b_G(M) \to \mathbb{D}^b_H(N) \) with right and left adjoints \( F^R, F^L: \mathbb{D}^b_H(N) \to \mathbb{D}^b_G(M) \) such that

1. There is an autoequivalence \( D_F = D \) of \( \mathbb{D}^b_G(M) \), called the \( \mathbb{P} \)-cotwist of \( F \), such that
   \[ F^R \circ F \simeq \text{id} \oplus D \oplus D^2 \oplus \cdots \oplus D^n. \]

2. Let \( \varepsilon: F \circ F^R \to \text{id} \) be the counit of the adjunction. The map
   \[ D \circ F^R \circ F \to F^R \circ F \circ F^R \circ F \xrightarrow{\varepsilon F} F^R \circ F, \]
   when written in the components
   \[ D \oplus D^2 \oplus \cdots \oplus D^n \oplus D^{n+1} \to \text{id} \oplus D \oplus D^2 \oplus \cdots \oplus D^n, \]
   is of the form
   \[
   \begin{pmatrix}
   * & * & \cdots & * & * \\
   1 & * & \cdots & * & * \\
   0 & 1 & \cdots & * & * \\
   \vdots & \vdots & \ddots & \vdots & \vdots \\
   0 & 0 & \cdots & 1 & * 
   \end{pmatrix}.
   \]

3. \( F^R \simeq D^n \circ F^L \). If \( \mathbb{D}^b_G(M) \) as well as \( \mathbb{D}^b_H(N) \) have Serre functors, this is equivalent to \( S_N \circ F \circ D^n \simeq F \circ S_M \).

A \( \mathbb{P}^1 \)-functor is the same as a split spherical functor. A general spherical functor is a FM transform \( F \) such that \( C := \text{cone} \circ \eta \to F^R \circ F \) is an autoequivalence and \( F^R \simeq C \circ F^L \). Here, \( \eta \) is the unit of the adjunction. The cone is well defined as a FM transform, since the natural transform \( \eta \) is induced by a morphism between the FM kernels; see \[AL12\]. This is the reason that we restrict ourself in the definition of spherical and \( \mathbb{P} \)-functors to Fourier–Mukai transforms between derived categories of coherent sheaves. More generally, one can work with dg-enhanced triangulated categories; see \[AL13\].

7.2.3 Notations and conventions

1. For \( E \) a complex we denote its \( q \)-th cohomology by \( \mathcal{H}^q(E) \) and set \( \mathcal{H}^*(E) := \oplus_{i \in \mathbb{Z}} \mathcal{H}^i(E)[-i] \).

2. For \( L \) a \( G \)-equivariant line bundle on a variety \( T \) we denote the tensor product functor by \( M_L := (\_ \otimes L): \mathbb{D}^b_G(T) \to \mathbb{D}^b_G(T) \). If we write \( \mathcal{O}_T \) as a \( G \)-sheaf we mean the structure sheaf equipped with its canonical linearisation.

3. The alternating representation \( a_n \) of the symmetric group \( \mathfrak{S}_n \) is the one-dimensional representation on which \( \sigma \in \mathfrak{S}_n \) acts by multiplication by \( \text{sgn} \). If \( \mathfrak{S}_n \) acts on a variety \( T \), we set \( M_{a_n} := M_{\mathcal{O}_T \otimes \mathbb{C}_n}: \mathbb{D}^b_{\mathfrak{S}_n}(T) \to \mathbb{D}^b_{\mathfrak{S}_n}(T) \).

4. For \( u \leq v \) positive integers, we use the notations \( [u, v] := \{u, u + 1, \ldots, v\} \subset \mathbb{N} \) and \( [v] := \{1, v\} = \{1, \ldots, v\} \subset \mathbb{N} \).
5. We set \([0] := \emptyset\).

6. For \(A, B \subset \mathbb{N}\) two finite subsets of the same cardinality \(|A| = |B|\) we let \(e: A \to B\) denote the unique strictly increasing bijection.

### 7.2.4 The Fourier–Mukai kernel

Let \(X\) be a smooth variety of arbitrary dimension \(d = \dim X\). In the following we will construct the functors \(H_{\ell,n}: \mathcal{D}^b_{\mathcal{G}_n}(X \times X^\ell) \to \mathcal{D}^b_{\mathcal{G}_{n+\ell}}(X^{n+\ell})\) for \(n, \ell \in \mathbb{N}\) with \(n \geq 2\). We consider \(\ell\) and \(n\) as fixed and mostly omit them in the notations. For \(i = 1, \ldots, \ell\) we set

\[
\text{Index}(i) := \{(I, J, \mu) \mid I \subset [\ell], |I| = i, J \subset [n+\ell], |J| = n+i, \mu: \tilde{I} \to \tilde{J} \text{ bijection}\}
\]

where \(\tilde{I} := [\ell] \setminus I\) and \(\tilde{J} := [n+\ell] \setminus J\) denote the complements of \(I\) and \(J\), respectively. For \((I, J, \mu) \in \text{Index}(i)\) we consider the subvariety \(\Gamma_{I,J,\mu} \subset X \times X^\ell \times X^{n+\ell}\) given by

\[
\Gamma_{I,J,\mu} := \{(x, x_1, \ldots, x_\ell, y_1, \ldots, y_{n+\ell}) \mid x = x_a \forall a \in I, b \in J, x_c = y_{\mu(c)} \forall c \in \tilde{I}\}.
\]

This subvariety is invariant under the action of the subgroup

\[
\mathcal{G}_I \times \mathcal{G}_{I,\mu} \times \mathcal{G}_J := \{((\sigma, \tau) \mid \sigma(I) = I, \sigma(J) = J, (\mu \circ \sigma)|_{\tilde{I}} = (\tau \circ \mu)|_{\tilde{J}}) \subset \mathcal{G}_\ell \times \mathcal{G}_{n+\ell}\}
\]

and thus \(O_{I,J,\mu} := O_{\Gamma_{I,J,\mu}}\) carries a canonical linearisation by this subgroup. Note that there is the isomorphism of groups \(\mathcal{G}_I \times \mathcal{G}_{I,\mu} \times \mathcal{G}_J \cong \mathcal{G}_I \times \mathcal{G}_{\ell-i} \times \mathcal{G}_{n+1}\) given by \((\sigma, \tau) \mapsto (\sigma|_I, \sigma|_{\tilde{I}}, \tau|_J)\).

Let \(\alpha_J\) denote the one-dimensional representation of \(\mathcal{G}_I \times \mathcal{G}_{I,\mu} \times \mathcal{G}_J\) on which the factor \(\mathcal{G}_I = \{\sigma = \text{id}\}\) acts by the sign of the permutations and the other factor \(\mathcal{G}_J \times \mathcal{G}_{I,\mu} = \{\tau|_I = \text{id}\}\) acts trivially. We set \(P(I, J, \mu) := O_{I,J,\mu} \otimes \alpha_J\) and

\[
P^i := \text{Inf}_{[i]} \mathcal{P}([i], [n+i], c) = \bigoplus_{\text{Index}(i)} \mathcal{P}(I, J, \mu).
\]

For \(c \in \tilde{I}\) we have \(\Gamma_{I,J,c,\mu(c),\mu|_{\{c\}}} \subset \Gamma_{I,J,\mu}\). This allows us to define a differential \(d^i: \mathcal{P}^i \to \mathcal{P}^{i+1}\) by letting the component \(\mathcal{P}(I, J, \mu) \to \mathcal{P}(I \cup \{c\}, J \cup \{\mu(c)\}, \mu|_{I \cup \{c\}})\) be \((-1)^{|\{c \in J \setminus b \leq \mu(c)\}|}\) times the map given by restriction of sections and setting all components which are not of this form to be zero. The resulting \(\mathcal{G}_I \times \mathcal{G}_{n+\ell}\)-equivariant complex

\[
P_{\ell,n} := P := (0 \to \mathcal{P}^0 \to \cdots \to \mathcal{P}^{\ell} \to 0) \subset \mathcal{D}^b_{\mathcal{G}_I \times \mathcal{G}_{n+\ell}}(X \times X^\ell \times X^{n+\ell})
\]

is the Fourier–Mukai kernel of our functor \(H_{\ell,n}\), that means

\[
H_{\ell,n} := \text{FM}_{\mathcal{P}_{\ell,n}}: \mathcal{D}^b_{\mathcal{G}_n}(X \times X^\ell) \to \mathcal{D}^b_{\mathcal{G}_{n+\ell}}(X^{n+\ell}).
\]

### 7.2.5 Adjoint kernels

Even though we do not assume that \(X\) is projective, since \(X \times X^\ell\) and \(X^{n+\ell}\) are smooth and \(\text{supp} \mathcal{P} = \text{supp} \mathcal{P}^0\) is projective over \(X \times X^\ell\) as well as over \(X^{n+\ell}\), the functor \(H_{\ell,n}\) has right and left adjoints \(H_{\ell,n}^R, H_{\ell,n}^L: \mathcal{D}^b_{\mathcal{G}_{n+\ell}}(X^{n+\ell}) \to \mathcal{D}^b_{\mathcal{G}_\ell}(X \times X^\ell)\) mapping to the bounded derived category. Their FM kernels are given by

\[
\mathcal{P}^R = \mathcal{P}^\vee \otimes (\omega_{X \times X^\ell} \otimes \mathcal{O}_{X^{n+\ell}})[(\ell + 1)d], \quad \mathcal{P}^L = \mathcal{P}^\vee \otimes (\mathcal{O}_{X \times X^\ell} \otimes \omega_{X^{n+\ell}})[(n + \ell)d]; \quad (7.9)
\]
see e.g. [Kuz06, Section 2.1]. Using this, we can go ahead and show that for $X$ a smooth surface the functor $H_{t,n}$ satisfies $H^R_{t,n} \cong \tilde{S}_X^{-1}H^L_{t,n}$, i.e. condition (iii) of a $\mathbb{P}^{n-1}$-functor with cotwist $\tilde{S}_X^{-1} := (\_ \otimes (\omega_X^{-1} \otimes \mathcal{O}_{X'})[-2])$. By (7.9) this amounts to the invariance of $\mathcal{P}^v$ under tensor product by $\omega_X^n \otimes \omega_X^{-1} \otimes \omega_X^{-1}_n$. This follows from the fact that

$$(\omega_X^n \otimes \omega_X^{-1} \otimes \omega_X^{-1}_n)|_{\Gamma_{I,J,\mu}} \cong \mathcal{O}_{I,J,\mu} \text{ for all } 0 \leq i \leq \ell \text{ and } (I, J, \mu) \in \text{Index}(i).$$

7.2.6 Description of the functor

For $I \subset [\ell]$ and $J \subset [n+\ell]$ with $|I| = i$ and $|J| = n+i$ there are the partial diagonals

$$X \times X^\ell \supset D_I := \{(x, x_1, \ldots, x_\ell) \mid x = x_\alpha \forall \alpha \in I\} \cong X \times X^{\ell-i},$$

$$X^{n+i} \supset \Delta_J := \{(y_1, \ldots, y_{n+i}) \mid y_\alpha = y_\beta \forall \alpha, \beta \in J\} \cong X \times X^{\ell-i},$$

and we denote the corresponding closed embeddings by $\iota_I : X \times X^{\ell-i} \to X \times X^\ell$ and $\delta_J : X \times X^{\ell-i} \to X^{n+i}$. We set $\iota_{[0]} = \iota_0 := \text{id} : X \times X^\ell \to X \times X^\ell$. The functor $H^i := H^i_{t,n} := \text{FM}_{p^i}$ is the composition

$$D^b_{\mathcal{E}_x}(X \times X^\ell) \xrightarrow{\text{Res}} D^b_{\mathcal{E}_x \times \mathcal{E}_{\ell-1}}(X \times X^\ell) \xrightarrow{\iota_I^*} D^b_{\mathcal{E}_x \times \mathcal{E}_{\ell-1}}(X \times X^{\ell-i})$$

and $\delta_{[n+i]}^* : D^b_{\mathcal{E}_{n+i} \times \mathcal{E}_{\ell-1}}(X^{n+i} \times X^{\ell-i}) \xrightarrow{\text{Inf}} D^b_{\mathcal{E}_{n+i} \times \mathcal{E}_{\ell}}(X^{n+i})$. For $i = 0$ this reduces to

$$D^b_{\mathcal{E}_x}(X \times X^\ell) \xrightarrow{\text{triv}} D^b_{\mathcal{E}_{n+\ell} \times \mathcal{E}_x}(X \times X^\ell) \xrightarrow{\text{M}_{a_\alpha}} D^b_{\mathcal{E}_{n+i} \times \mathcal{E}_x}(X \times X^\ell) \xrightarrow{\text{Inf}} D^b_{\mathcal{E}_{n+i} \times \mathcal{E}_x}(X^{n+i}).$$

Note that (7.11) differs by the tensor product $\text{M}_{a_\alpha}$ with the alternating representation from the suggestion (7.2) from the introduction. Using slightly shortened notation, the functor $H^i$ is on the level of objects given by

$$H^i : E \mapsto \bigoplus_{J \subset [n+i], \#J = n+i} \delta_{J}^* \left( a_J \otimes \iota_{[i]}^*(E)^{\mathcal{E}_{\ell}} \right).$$

The right-adjoint $H^i_R : D^b_{\mathcal{E}_{n+i}}(X^{n+i}) \to D^b_{\mathcal{E}_x}(X^\ell)$ is given by the composition

$$D^b_{\mathcal{E}_x}(X \times X^\ell) \xrightarrow{\text{Inf}} D^b_{\mathcal{E}_x \times \mathcal{E}_{\ell-1}}(X \times X^\ell) \xrightarrow{\iota_I^*} D^b_{\mathcal{E}_x \times \mathcal{E}_{\ell-1}}(X \times X^{\ell-i})$$

$$\xrightarrow{\text{triv}} D^b_{\mathcal{E}_{\ell-1}}(X \times X^{\ell-i}) \xrightarrow{\delta_{[n+i]}^*} D^b_{\mathcal{E}_{n+i} \times \mathcal{E}_{\ell-1}}(X \times X^{\ell-i}) \xrightarrow{\text{Res}} D^b_{\mathcal{E}_{n+i} \times \mathcal{E}_{\ell}}(X^{n+i}).$$

which means on the level of objects $F \in D^b_{\mathcal{E}_{n+i}}(X^{n+i})$ that

$$H^i_R : E \mapsto \bigoplus_{I \subset [\ell], \#I = i} \iota_{I}^* \left( a_{[n+i]} \otimes \delta_{[n+i]}^!(F)^{\mathcal{E}_{[n+i]}} \right).$$
7.3 Techniques and examples

7.3.1 Derived intersections

Given a vector bundle $E$ of rank $c$ on a variety $Z$ we write $\wedge^* E := \oplus_{i=0}^c \wedge^i E[-i]$ and $\wedge^{-*} E := \oplus_{i=0}^c \wedge^i E[i]$ as objects in $\mathcal{D}^b(Z)$.

**Theorem 7.3.1 ([AC12]).** Let $\iota: Z \hookrightarrow M$ be a regular embedding of codimension $c$ such that the normal bundle sequence $0 \rightarrow T_Z \rightarrow T_{M|Z} \rightarrow N_\iota \rightarrow 0$ splits. Then there is an isomorphism

$$\iota^* \iota_*(\_ \otimes \wedge^{-*} N_\iota^\vee)$$

of endofunctors of $\mathcal{D}^b(Z)$.

Recall that the right-adjoint of $\iota^*$ is given by $\iota^! = M_\omega_\iota \circ \iota^*[\text{codim} \iota]$ where $\omega_\iota = \wedge^\text{codim} \iota N_\iota$; see [Har66, Corollary III 7.3]. We have $\wedge^{-*} N_\iota^\vee \otimes \omega_\iota[\text{codim} \iota] \cong \wedge^* N_\iota$.

**Corollary 7.3.2.** Under the assumptions of the previous theorem, there is an isomorphism

$$\iota^! \iota_*(\_ \otimes \wedge^* N_\iota).$$

In particular, the derived self-intersection $\iota^* \iota_* O_Z$ of $Z$ in $M$ is given by $\iota^* \iota_* O_Z = \wedge^{-*} N_\iota^\vee$. Similar results for derived intersections, i.e. for $\iota_2^* \iota_{1*} O_{Z_2}$ when $\iota_1: Z_1 \rightarrow M$, $\iota_2: Z_2 \rightarrow M$ are two different closed embeddings, are proven in [Gri13]. However, we will always be in the following situation where Theorem 7.3.1 is sufficient. Assume that there is a diagram

\[ T = Z_1 \cap Z_2 \]

\[ \xymatrix{ & Z_2 \ar[dl]_u \ar[dr]^r & \ar[d]^t & M \ar[dl]^u \ar[dr]^r & \ar[dl]_{\iota_2} & \ar[d]_{\iota_1} & \ar[u]_s \ar[dl]^u \ar[dr]^r & \ar[d]_{\iota_1} & \ar[u]_s & \ar[dl]^u \ar[dr]^r & \ar[d]_{\iota_1} & \ar[u]_s & \ar[dl]^u \ar[dr]^r & \ar[d]_{\iota_1} & \ar[u]_s } \]

where all the maps are regular closed embeddings, $t$ has a splitting normal bundle sequence, and the intersection of $Z_1$ and $Z_2$ inside of $W$ is transversal.

**Lemma 7.3.3.** Under the above assumptions there is the isomorphism of functors

$$\iota_2^* \iota_{1*}(\_ \otimes \wedge^{-*} N_\iota^\vee).$$

In particular, $\iota_2^* (\iota_{1*} O_{Z_1}) \cong \iota_*(\wedge^{-*} N_\iota^\vee)$.

**Proof.** Indeed, we have

$$\iota_2^* \iota_{1*} \cong r^* t^* t_\iota s_* \cong r^* (s_* (\_ \otimes \wedge^{-*} N_\iota^\vee)) \cong r^* s_* (\_ \otimes \wedge^{-*} N_\iota^\vee_{Z_2} \cong \iota_* v_* (\_ \otimes \wedge^{-*} N_\iota^\vee_{Z_2} \cong \iota_* v_* (\_ \otimes \wedge^{-*} N_\iota^\vee_{Z_2})$$

where the prior to last isomorphism is the base change theorem [Kuz06, Corollary 2.27].
Corollary 7.3.4. Under the same assumptions we have

\[ i_2^! u_{1*}(\_ \cong \approx u_*(v^*(\_ \otimes \wedge^* N_{W,T}^\vee) \otimes \omega_2[-\text{codim} i_2] \cong u_*(v^*(\_ \otimes \wedge^* N_{W,T} \otimes \omega_2)[-\text{codim} v]. \]

In particular, \( i_2^! (\iota_1^* O_{Z_1}) \cong u_*(\wedge^* N_{W,T}^\vee) \otimes \omega_2[-\text{codim} i_2] \cong u_*(\wedge^* N_{W,T} \otimes \omega_2)[-\text{codim} v]. \)

Note that by Grothendieck duality \( i_2^* \mathcal{H}^p (i_2^! \iota_{1*} O_{Z_1}) \cong \mathcal{E}xt_{\mathcal{O}_M}^p (O_{Z_2}, O_{Z_1}). \)

Remark 7.3.5. In the above situation consider in addition \( Z_2 \subset W'' \subset W \) such that \( w' : W' \to W \) is a regular embedding and \( W' \) and \( Z_1 \) intersect transversally. We set \( Z_1' = W' \cap Z_1 \). We also consider \( Z_1 \subset W'' \subset W \) such that \( w'' : W'' \to W \) is a regular embedding and \( W'' \) and \( Z_2 \) intersect transversally in \( Z_2'' = W'' \cap Z_2 \). So we have the two diagrams of closed embeddings

\[ \begin{array}{cc}
\begin{array}{ccc}
Z_2 & \xrightarrow{id} & Z_2 \\
\downarrow & & \downarrow \\
W' & \xrightarrow{\iota_2} & W \\
\downarrow & & \downarrow \\
Z_1 & \xrightarrow{r} & Z_1
\end{array}
& \quad &
\begin{array}{ccc}
Z_2'' & \xrightarrow{z''} & Z_2 \\
\downarrow & & \downarrow \\
W'' & \xrightarrow{\iota_2} & W \\
\downarrow & & \downarrow \\
Z_1 & \xrightarrow{r} & Z_1
\end{array}
\end{array} \]

We set \( \iota_1' = \iota_1 \circ z', \iota_2'' = \iota_2 \circ z'' \), and \( t'' = t \circ w'' \). The restriction map \( \iota_1^* O_{Z_1} \to \iota_1'^* O_{Z_1'} \) induces for every \( q = 0, \ldots, \text{codim}(t) \) the map

\[ u_*(\wedge^q N_{W,T}^\vee) \otimes \omega_2 \cong \mathcal{E}xt_{\mathcal{O}_M}^{\text{codim}(i_2)-q} (O_{Z_2}, O_{Z_1}) \to \mathcal{E}xt_{\mathcal{O}_M}^{\text{codim}(i_2)-q} (O_{Z_2'}, O_{Z_1'}) \cong u_*(\wedge^q N_{W,T}^\vee) \otimes \omega_2. \]

As one can check locally using the Koszul resolutions this map is given by the \( q \)-th wedge power of the canonical map \( N_{W,T}^\vee \to N_{W,T}^\vee \). Similarly, for \( q = 0, \ldots, \text{codim}(t) \) the induced map

\[ u_*(\wedge^q N_{W,T} \otimes \omega_2) \cong \mathcal{E}xt_{\mathcal{O}_M}^{\text{codim}(i_2)+q} (O_{Z_2'}, O_{Z_1'}) \to \mathcal{E}xt_{\mathcal{O}_M}^{\text{codim}(i_2)+q} (O_{Z_2}, O_{Z_1}) \cong u_*(\wedge^q N_{W,T} \otimes \omega_2) \]

given by the \( q \)-th wedge power of the canonical map \( N_{W,T} \to N_{W,T} \).

Remark 7.3.6. Let \( G \) be a finite group acting on \( M \) such that all the subvarieties occurring above are invariant under this action. Then all the normal bundles carry a canonically induced \( G \)-linearisation. All the results of this subsection continue to hold as isomorphisms in the (derived) categories of \( G \)-equivariant sheaves when considering the normal bundles as \( G \)-bundles equipped with the canonical linearisations; compare [LH09, Section 28].

7.3.2 Partial diagonals and the standard representation

Let \( I \) be a finite set of cardinality at least 2. The standard representation \( g_I \) of the symmetric group \( S_I \) can be considered either as the subrepresentation \( g_I \subset C^I \) of the permutation representation consisting of all vectors whose components add up to zero or as the quotient \( g_I = C^I / C \) by the one-dimensional subspace of invariants. For \( I \subset I' \) the first point of view gives a canonical \( S_I \)-equivariant inclusion \( g_I \to g_{I'} \) while the second one gives a canonical \( S_I \)-equivariant surjection \( g_{I'} \to g_I \). For \( X \) a smooth variety and \( \delta_n : X \to X^n \) the embedding of the small diagonal there is the \( S_n \)-equivariant isomorphism \( N_{\delta_n} \cong T_X \otimes g_n \); see [6, Section
3]. More generally, for \( I \subset [n] \) the normal bundle of the partial diagonal \( \Delta_I \cong X \times X^I \) is as a \( \mathcal{S}_I \)-bundle given by

\[
N_{\delta_I} \cong (T_X \otimes \varrho_I) \boxtimes \mathcal{O}_{X^I}, \quad N_{N_{\delta_I}}^\vee \cong (\Omega^I_X \otimes \varrho_I) \boxtimes \mathcal{O}_{X^I}.
\]  

(7.16)

Furthermore, the normal bundle sequence of \( \delta_I \) splits since \( \Delta_I \) is the fixed point locus of the \( \mathcal{S}_I \)-action on \( X^n \); see [AC12, Section 1.20].

**Remark 7.3.7.** For \( I \subset I' \subset [n] \), the embedding \( \Delta_{I'} \to \Delta_I \) induces maps \( N_{\delta_{I'}} \to N_{\delta_I|\Delta_{I'}} \) and \( N_{N_{\delta_{I'}}}^\vee \to N_{N_{\delta_I}}^\vee \). Under the isomorphisms (7.16) they are given by the canonical maps \( \varrho_{I'} \to \varrho_I \) and \( \varrho_I \to \varrho_I \), respectively.

For \( m \geq 2 \) and \( X \) a smooth variety of dimension \( d \), we set

\[
\Lambda_m^*(X) := \left( \wedge^*(T_X \otimes \varrho_m) \right)^\oplus_m = \bigoplus_{i=0}^{(m-1)d} \left( \wedge^i(T_X \otimes \varrho_m) \right)^\oplus_m [-i].
\]

(7.17)

**Lemma 7.3.8.**

\[
\Lambda_m^*(X) = \begin{cases} 
\mathcal{O}_X[0] & \text{for } X \text{ a curve,} \\
\mathcal{O}_X[0] \oplus \omega_X^{-1}[-2] \oplus \cdots \oplus \omega_X^{-(m-1)}[-2(m-1)] & \text{for } X \text{ a surface.}
\end{cases}
\]

**Proof.** For the surface case see [Sca09, Lemma B.5] and [6, Corollary 3.5]. Since \( \wedge^0(T_X \otimes \varrho_m) = \mathcal{O}_X \) is equipped with the trivial \( \mathcal{S}_m \)-action, we only have to show that \( \wedge^i(T_X \otimes \varrho_m) \) has no invariants for \( i \geq 1 \) in the case that \( X \) is a curve. For this it is sufficient to consider the fibres which are given by \( \wedge^i \varrho_m \). By [FH91, Proposition 2.12] the representations \( \wedge^i \varrho_m \) are irreducible. They are non-trivial for \( i \geq 1 \) hence their invariants vanish.

**Remark 7.3.9.** For \( d = \dim X \geq 3 \) also vector bundles of higher rank occur as direct summands of \( \Lambda_m^*(X) \). For example, for \( m = 2 \) we have

\[
\Lambda_2^*(X) \cong \bigoplus_{0 \leq k \leq d/2} \wedge^{2k} T_X[-2k].
\]

**Remark 7.3.10.** For \( I \subset [n] \) of cardinality \( m := |I| \geq 2 \) consider the functor \( G = \delta_{I^c} \circ \text{triv} \) as well as \( G^R \circ G \), i.e. the composition

\[
D^b_{\mathcal{E}_I}(X \times X^I) \xrightarrow{\text{triv}} D^b_{\mathcal{E}_I \times \mathcal{E}_I}(X \times X^I) \xrightarrow{\delta_{I^c}} D^b_{\mathcal{E}_I \times \mathcal{E}_I}(X^n) \xrightarrow{\delta_I} D^b_{\mathcal{E}_I}(X \times X^I).
\]

Let \( \text{pr}_X : X \times X^I \to X \) be the projection to the first factor. Corollary 7.3.2 together with Lemma 7.3.8 give

\[
G^R \circ G \cong (\_ \otimes \text{pr}_X^*) \Lambda^*_m(X) \cong \begin{cases} 
\text{id} & \text{for } X \text{ a curve,} \\
\mathcal{S}_X^{-[0,m-1]} := \text{id} \oplus \mathcal{S}_X^{-1} \oplus \cdots \oplus \mathcal{S}_X^{-(m-1)} & \text{for } X \text{ a surface.}
\end{cases}
\]

(7.18)

Here, \( \mathcal{S}_X := (\_ \otimes (\omega_X \boxtimes \mathcal{O}_{X^I})[-2] \) for a (not necessarily projective) smooth surface \( X \).
7.3.3 The case $\ell = 0$

In the special case that $I = [n]$ we have $G = M_{n,n} \circ H_{0,n}$ and (7.18) gives

$$H_{0,n} \circ H_{0,n} \cong \begin{cases} \text{id} & \text{for } X \text{ a curve}, \\ S_X^{-[0,n-1]} := \text{id} \oplus S_X^{-1} \oplus \cdots \oplus S_X^{-(n-1)} & \text{for } X \text{ a surface}. \end{cases}$$

This proves the case $\ell = 0$ of Proposition A(i) and most of Theorem C (for the proof that condition (ii) of a $P_{n-1}$-functor holds for $H_{0,n}$ in the surface case, see [6, Section 3]).

7.3.4 The approach for general $\ell$

There is the commutative diagram

$$\begin{array}{ccccccc}
\mathcal{P}^{iR} \ast \mathcal{P} & \longrightarrow & \mathcal{P}^{iR} \ast \mathcal{P}^0 & \longrightarrow & \cdots & \longrightarrow & \mathcal{P}^{iR} \ast \mathcal{P}^\ell \\
\downarrow & & \downarrow & & \cdots & & \downarrow \\
\vdots & & \vdots & & \cdots & & \vdots \\
\mathcal{P}^0 \ast \mathcal{P} & \longrightarrow & \mathcal{P}^0 \ast \mathcal{P}^0 & \longrightarrow & \cdots & \longrightarrow & \mathcal{P}^0 \ast \mathcal{P}^\ell \\
\downarrow & & \downarrow & & \cdots & & \downarrow \\
\vdots & & \vdots & & \cdots & & \vdots \\
\mathcal{P}^R \ast \mathcal{P} & \longrightarrow & \mathcal{P}^R \ast \mathcal{P}^0 & \longrightarrow & \cdots & \longrightarrow & \mathcal{P}^R \ast \mathcal{P}^\ell \\
\downarrow & & \downarrow & & \cdots & & \downarrow \\
\vdots & & \vdots & & \cdots & & \vdots \\
\end{array}$$

(7.19)

where the $\mathcal{P}^{iR} \ast \mathcal{P}$ and $\mathcal{P}^R \ast \mathcal{P}^j$ are the left and right convolutions of the rows and columns, respectively. That means in particular that $\mathcal{P}^R \ast \mathcal{P}^j$ can be written as a multiple cone

$$\text{cone}(\ldots \text{cone}(\text{cone}(\mathcal{P}^{iR} \ast \mathcal{P}^j \to \mathcal{P}^{(\ell-1)R} \ast \mathcal{P}^j) \to \mathcal{P}^{(\ell-2)R} \ast \mathcal{P}^j) \ldots) \to \mathcal{P}^{0R} \ast \mathcal{P}^j);$$

see e.g. [Kaw04, Section 2] for the notion of convolutions in triangulated categories. The strategy of the proof of Proposition A and Theorem C that will be given in Section 7.4 is to start with the computation of the $\mathcal{P}^{iR} \ast \mathcal{P}^j$, then use the results to compute the $\mathcal{P}^R \ast \mathcal{P}^j$, and finally deduce the desired formulae for $\mathcal{P}^R \ast \mathcal{P}$.

In the following subsections we will do the computation of the $\mathcal{P}^{iR} \ast \mathcal{P}^j$ in the case $\ell = 1$ (and some of it for $\ell = 2$) on the level of the functors. That means that we will compute the compositions $H_{i,n}^R H_{j,n}^j$. We will see that the undesired terms (compare (7.4)) are of a form which give them a good chance to cancel out when passing to $H_{i,n}^R H_{j,n}^i$. But we will not compute the induced maps $H_{i,n}^{Riem} H_{j,n}^j \to H_{i,n}^{Riem} H_{j,n}^{i+1}$ and $H_{i,n}^R H_{j,n}^j \to H_{i,n}^{(i-1)Riem} H_{j,n}^j$. Hence, we will not see that the terms really cancel until Section 7.4 where the computations are performed for general $\ell$ on the level of the FM kernels.

7.3.5 Invariants of inflations

For the computation of the invariants we will use the following principle; compare [Dan01, Lemma 2.2] and [Sca09, Remark 2.4.2]. Let $M$ be a variety with an action of a finite group $G$. Let $E = (E, \lambda) \in D^b_G(M)$ such that $E = \oplus_J E_j$ in $D^b(M)$ for some finite index set $I$. Let us assume that there is an action of $G$ on $I$ such that $\lambda_g(E_i) = g^* E_{g(i)}$ for all $i \in I$. We say
that the $G$-action on $I$ is induced by the $G$-linearisation of $E$. For example, the $\mathcal{S}_\ell \times \mathcal{S}_{\ell+n}$-linearisation of $\mathcal{P}^i$ induces the action $\sigma \times \varpi : (I, J, \mu) \mapsto (\sigma(I), \varpi(J), \tau \circ \mu \circ \sigma^{-1})$ on the index set $\text{Index}(i)$. We denote $E_i$ together with the $G_i$-linearisation $(\lambda_g E_i)_{g \in G_i}$ by $\varepsilon_i \in D_{G_i}^b(M)$ where $G_i = \text{Stab}_G(i)$. The induced action of $G$ on $I$ is transitive if and only if $\mathcal{E} \cong \text{Inf}_{G_i}^G \varepsilon_i$ for any $i \in I$; see [BL94, Section 8.2].

In the case that $G$ acts trivially on $M$, for every $i \in I$ the projection $E \to E_i$ induces the isomorphism $\mathcal{E}^G \cong \varepsilon_i^{G_i}$. The inverse is given by $s \mapsto \oplus_{[g] \in G/G_i} \lambda_g \varepsilon_i$. Let the action of $G$ on $I$ be not transitive with $i_1, \ldots, i_k$ being a system of representatives of the $G$-orbits. Then $\mathcal{E} \cong \text{Inf}_{G_{i_1}}^G \varepsilon_1 \oplus \cdots \oplus \text{Inf}_{G_{i_k}}^G \varepsilon_k$ and

$$\mathcal{E}^G \cong \varepsilon_1^{G_{i_1}} \oplus \cdots \oplus \varepsilon_k^{G_{i_k}}. \tag{7.20}$$

### 7.3.6 The case $\ell = 1$

Let $\ell = 1$ and $n > 1$. We aim to compute $H^0 R \circ H : D^b(X \times X) \to D^b(X \times X)$ where $H := H_{1,n}$ using the descriptions (7.12) and (7.14) of $H^2$ and $H^{-2}$. We have

$$H^0 R H^0(E) \cong \left[ a[n] \otimes \delta[n] \left( \bigoplus_{a \in [n+1]} \delta[n+1] \setminus \{a\} \ast (E \otimes a[n+1] \setminus \{a\}) \right) \right] \mathcal{S}[n]$$

for $E \in D^b(X \times X)$.

For $\sigma \in \mathcal{S}[n]$, the $\mathcal{S}[n]$-linearisation of $\bigoplus_{a \in [n+1]} \delta[n] \delta[n+1] \setminus \{a\} \ast (E \otimes a[n+1] \setminus \{a\})$ maps the summand $\delta[n] \delta[n+1] \setminus \{a\} \ast (E \otimes a[n+1] \setminus \{a\})$ to $\delta[n] \delta[n+1] \setminus \{\sigma(a)\} \ast (E \otimes a[n+1] \setminus \{\sigma(a)\})$. Thus, the induced action on the index set $[n+1]$ is given by $a \mapsto \sigma(a)$. Hence there are two $\mathcal{S}[n]$-orbits, namely $[n]$ and $\{n+1\}$. We have Stab$\mathcal{S}[n](n+1) = \mathcal{S}[n]$ and Stab$\mathcal{S}[n](n) = \mathcal{S}[n-1]$. As explained in Section 7.3.5 it follows that

$$H^0 R H^0(E) \cong \delta[n] \delta[n+1] \ast (E) \mathcal{S}[n] \oplus \delta[n+1] \ast (E) \mathcal{S}[n-1].$$

The first direct summand equals $E \otimes \text{pr}_1\ast \Lambda^n(X)$ by (7.18).

**Convention 7.3.11.** For $\{b\} \subset [m]$ a set with one element we set $\Delta_{\{b\}} = X^m$. Furthermore, we set $\Lambda^1(X) = O_X(0)$. These notations occur in this subsection in the case that $n = 2$ and later in the more general case that $n = \ell + 1$.

For the computation of the second summand, consider the commutative diagram of closed embeddings

$$\begin{array}{ccc}
\Delta_{[n+1]} & \xrightarrow{u} & \Delta_{[n]} \\
\downarrow \quad & & \downarrow \\
\Delta_{[n-1]} & \xrightarrow{\delta_{[n-1]}} & X^{n+1}.
\end{array}$$

It fulfills the properties of diagram (7.15) which means that $\Delta_{[n+1]} = \Delta_{[n]} \cap \Delta_{[n-1] \cup \{n+1\}}$ and that this intersection is transversal inside $\Delta_{[n-1]}$. Furthermore, the normal bundle sequence of $\delta_{[n-1]}$ splits; see Section 7.3.2. This allows us to apply Corollary 7.3.4 to get

$$\delta_{[n]} \ast (E) \cong u_*(v^* \otimes \wedge^* N_{\Delta_{[n-1]} \cap \Delta_{[n+1]}} \otimes \omega_v)[- \text{codim } v]. \tag{7.21}$$
Under the isomorphisms $\Delta_{[n+1]} \cong X$ and $\Delta_{[n]} \cong X \times X \cong \Delta_{[n-1] \cup \{n+1\}}$, the embeddings $u$ and $v$ equal the diagonal embedding $\imath: X \to X \times X$. Thus, $\text{codim } v = \dim X = d$ and $\omega_v \cong \Lambda^d N_v \cong \omega_X^1$. It follows together with (7.21) that after taking $\mathcal{G}_{[n-1]}$-invariants in (7.21) we get $\delta_{[n]}^* \delta_{[n-1] \cup \{n+1\}^*}(E)^{\mathcal{G}_{[n-1]}} \cong t_*(\imath^* (E) \otimes \Lambda_{n-1}^* (X) \otimes \omega_X^{-1})[-d]$. In summary,

$$H^{0R}H^0(E) \cong E \otimes \text{pr}_1^* \Lambda_{n}^* (X) \oplus t_*(\imath^* (E) \otimes \Lambda_{n-1}^* (X) \otimes \omega_X^{-1})[-d]. \quad (7.22)$$

The computation of the other three functor compositions is easier. Note that we have $\delta_{[n+1]} = \delta_{[n]} \circ u$ and $u^! \cong u^*(\imath) \otimes \omega_X^{-1}[-d]$. Hence,

$$H^{0R}H^1(E) \cong \left[ a_{[n]} \otimes \delta_{[n]}^* \delta_{[n+1]}^* (\imath^* E \otimes a_{[n+1]}) \right]^{\mathcal{G}_{[n]}} \cong \delta_{[n]}^* \delta_{[n]}^* (E)^{\mathcal{G}_{[n]}} \quad (7.23)$$

$$H^{1R}H^0(E) \cong t_* \left[ a_{[n]} \otimes \delta_{[n+1]}^* \left( \bigoplus_{a=1}^{n+1} \delta_{[n+1]\{a\}}^* (E \otimes a_{[n+1]\{a\}}) \right) \right]^{\mathcal{G}_{[n+1]}} \quad (7.24)$$

$$H^{1R}H^1(E) \cong t_* \left[ \delta_{[n+1]}^* \delta_{[n+1]}^* (E)^{\mathcal{G}_{[n]}} \right] \cong t_* \left( \imath^* (E) \otimes \Lambda_{n}^* (X) \otimes \omega_X^{-1} \right)[-d]. \quad (7.25)$$

Let $X = C$ be a smooth curve. By Lemma 7.3.8 we have $\Lambda_{m}^* (C) = \mathcal{O}_C [0]$ for all $m \geq 1$. Plugging this into (7.22), (7.23), (7.24), and (7.25), we get

$$H^{1R}H^0 \longrightarrow H^{1R}H^1 \quad \cong \quad t_* (\imath^* (\_ \otimes \omega_C^{-1})[-1] \longrightarrow t_* \imath^* \quad (7.26)$$

$$H^{1R}H^0 \longrightarrow H^{1R}H^1 \quad \text{id} \oplus t_* (\imath^* (\_ \otimes \omega_C^{-1})[-1] \longrightarrow t_* \imath^* \quad (7.26)$$

We will see in Section 7.4.6 that the right-hand vertical map of this diagram as well as the component $t_* (\imath^* (\_ \otimes \omega_C^{-1})[-1] \to t_* (\imath^* (\_ \otimes \omega_C^{-1})[-1]$ of the left-hand vertical map are isomorphisms. Thus, by taking cones in the diagram (7.19) enlarging (7.26) we get $H^{R}H^0 \cong \text{id}$ and $H^{R}H^1 = 0$. Considering the triangle $H^{R}H \to H^{R}H^0 \to H^{R}H^1$ shows $H^{R}H = \text{id}$, i.e. Proposition A(i) in the case $\ell = 1$.

For $X$ a smooth surface we have $(\_ \otimes \Lambda_{m}^* (X) \cong S^{-[0,m-1]}_X$, see (7.18). This gives

$$H^{1R}H^0 \longrightarrow H^{1R}H^1 \quad t_* S^{-[1,n]}_X \otimes \longrightarrow t_* S^{-[0,n]}_X \quad (7.26)$$

$$H^{1R}H^0 \longrightarrow H^{1R}H^1 \quad \cong \quad S^{-[0,n-1]}_X \oplus t_* S^{-[1,n-1]}_X \longrightarrow t_* S^{-[0,n-1]}_X \quad (7.26)$$

where $S^{-1}_X = (\_ \otimes \omega_X^{-1}[-2]$. Again, we will see later that all components of the maps in the diagram of the form $t_* S^{-k}_X \otimes \longrightarrow t_* S^{-k}_X$ are isomorphisms which gives by taking cones

$$H^{R}H^0 \cong S^{-[0,n-1]}_X \oplus t_* S^{-n}_X \otimes [1] \quad , \quad H^{R}H^1 \cong t_* S^{-n}_X \otimes [1]$$

and finally $H^{R}H^0 \cong S^{-[0,n-1]}_X$ as claimed in Theorem C.
7.3.7 Orthogonality in the curve case

We want to compute that $H^R_{1,n}H_{0,n+1} = 0$ for $X = C$ a curve which is one instance of Proposition A (ii). We have

\[ H^R_{1,n}H_{0,n+1}(E) \cong \tau_*(\delta_{[n+1]}^\delta(E))^{\oplus n+1} \cong \tau_*(E \otimes \Lambda_{n+1}^n(X)) \quad (7.27) \]

\[ H^R_{1,n}H_{0,n+1}(E) \cong \delta_{[n]}^\delta(E)^{\oplus n} \cong \delta_{[n]}^\delta u_*(E)^{\oplus n} \cong \tau_*(E \otimes \Lambda_n^0(X)) \quad (7.28) \]

For $X = C$ a curve this gives $H^R_{1,n}H_{0,n+1} \cong \text{id}$ and $H^R_{1,n}H_{0,n+1} \cong \text{id}$. By the exact triangle

\[ H^R_{1,n}H_{0,n+1} \rightarrow H^R_{1,n}H_{0,n+1} \rightarrow H^R_{1,n}H_{0,n+1} \quad (7.29) \]

we get the vanishing $H^R_{1,n}H_{0,n+1} = 0$.

7.3.8 Non-orthogonality in the surface case

For $X$ a smooth surface we have $H^R_{1,n}H_{0,n+1} \cong \tau_* S_{X}^{-[0,n]}$ and $H^R_{1,n}H_{0,n+1} \cong \tau_* S_{X}^{-[0,n-1]}$ by (7.27) and (7.28). Again, all the components $\tau_0 S_{X}^{-k} \rightarrow \tau_0 S_{X}^{-k}$ of the induced map $H^R_{1,n}H_{0,n+1} \rightarrow H^R_{1,n}H_{0,n+1}$ are isomorphisms for $k = 0, \ldots, n - 1$. Thus,

\[ H^R_{1,n}H_{0,n+1} \cong \tau_* S_{X}^{-[1]} \quad (7.30) \]

by triangle (7.29). Let $E, F \in D^b(X)$. As in Section 7.1.3 we set $H_{0,n+1}(F) = H_{0,n+1} \circ I_F$ and $H_{1,n}(E) = H_{1,n} \circ I_E$. Note that the domain of $H_{0,n+1}(F)$ is $D^b(pt)$ and the functor is given by sending the generator $C[0]$ to $\delta_{[n+1]}(F)$. By (7.30) we have

\[ H_{1,n}(E)^R H_{0,n+1}(F)(C[0]) \cong \text{pr}_{2*} \mathcal{H}om(E \otimes O_X, \tau_0 F \otimes \omega_{X}^n) [-2n + 1] \]

\[ \cong \mathcal{H}om(E, F \otimes \omega_{X}^n)[-2n + 1]. \]

That seems not to fit well with the commutator relation (7.5) which states $q_{1,-n}(\alpha) \circ q_{0,n+1}(\beta) = 0$.

7.3.9 The case $\ell = 2, n = 2$

Set $H = H_{2,2}: D^b_{\mathcal{O}_2}(X \times X^2) \rightarrow D^b_{\mathcal{O}_4}(X^4)$. We consider this case in order to illustrate why the assumption that $n > \ell$ is necessary for Proposition A and Theorem C. We have

\[ H^R H^0(E) \cong \big[ a_{[2]} \otimes \delta_{[2]} \big] \big( \bigoplus_{J \subset [4], |J| = 2} \delta_{J*}(E \otimes a_J) \big)^{\oplus [2]} \].

For $J = [2]$ we get the direct summand $\delta_{[2]}^\delta(E)^{\oplus [2]} \cong E \otimes \text{pr}_{X}^* \Lambda_n^0(X)$ which has the shape of a fully faithful functor for $X$ a curve and the shape of a $\mathbb{P}^1$-functor with cotwist $S_X^{-1}$ for $X$ a surface. For $J = [3, 4]$ we consider the diagram

\[ \Delta_{[2]} \cap \Delta_{[3, 4]} \xrightarrow{u} \Delta_{[2]} \]

\[ \downarrow v \quad \delta_{[2]} \downarrow \]

\[ \Delta_{[3, 4]} \xrightarrow{\delta_{[3, 4]}} X^4 \]
which is a transversal intersection. Under the isomorphism $\Delta_{[2]} \cong X \times X^2$ the subvariety $X \times X \cong \Delta_{[2]} \cap \Delta_{[3,4]} \subset \Delta_{[2]}$ equals $X \times \Delta_X$. Thus, for an appropriate choice of $E$ the direct summand $\left[ a_{[2]} \otimes \delta_{[2]}(E \otimes a_{[3,4]}) \right]_{[\theta]}$ of $H^0 R \mathcal{H}^0(E)$ is supported on the whole $X \times \Delta_X$.

On the other hand one can see easily that all direct summands of $H^i R \mathcal{H}^j(E)$ for $(i, j) \neq (0, 0)$ are supported on one of the subvarieties $D_1$, $D_2$ or $D_{[2]}$ of $X \times X^2$, neither of them containing $X \times \Delta_X$. It follows that the direct summand $\left[ a_{[2]} \otimes \delta_{[2]}(E \otimes a_{[3,4]}) \right]_{[\theta]}$ of $H^0 R \mathcal{H}^0$ survives taking the multiple cones in the diagram (7.19) which prevents $H^R \mathcal{H}$ from being isomorphic to id or $S_X^{-[0,1]}$.

## 7.4 Proof of the main results

Let $X$ be a smooth variety of dimension $d := \dim X$. Let furthermore $\ell, n \in \mathbb{N}$ with $n > \max\{\ell, 1\}$ and consider $\mathcal{P} = \mathcal{P}_{\ell, n}$; see Section 7.2.4. We will compute the convolution products $\mathcal{P}^R \star \mathcal{P}^I$.

In the case that $X$ is a curve or a surface that will lead to formulae for $\mathcal{P}^R \star \mathcal{P}^I$.

### 7.4.1 Computation of the direct summands

Let $(I_1, J_1, \mu_1) \in \text{Index}(j)$ and $(I_2, J_2, \mu_2) \in \text{Index}(i)$. We set $K_1 := I_1 \cup \mu_1^{-1}(J_2)$, $K_2 := I_2 \cup \mu_2^{-1}(J_1) \subset [\ell]$ and $\mu := \mu_2^{-1} \circ \mu_1 |_{K_1}$ as a bijection between $K_1 = [\ell] \setminus K_1$ and $\bar{K}_2 = [\ell] \setminus K_2$. Furthermore, let $\Gamma_{K_1, K_2, \mu} \subset X \times X^\ell \times X \times X^\ell$ be the subvariety

$$\Gamma_{K_1, K_2, \mu} := \{(x, x_1, \ldots, x_\ell, z, z_1, \ldots, z_\ell) \mid x = x_a = z_b = z \forall a \in K_1, b \in K_2, x_c = z_{\mu(c)} \forall c \in \bar{K}_1\}.$$ 

Consider the diagram

$$
\begin{array}{c}
\begin{tikzcd}
X \times X^\ell \times \Gamma_{I_2, J_2, \mu_2} \ar[rr]^u \ar[dr] & & X \times X^\ell \times \Delta_{J_1 \cap J_2} \times X \times X^\ell \ar[rr]^t \ar[dr] & & X \times X^\ell \times X^\ell \times X \times X^\ell \\
\Gamma_{I_1, J_1, \mu_1} \times X \times X^\ell \ar[rrrrr]^s \ar[d, \cong, \pi_{13}] & & & & & X \times X^\ell \times X \times X^\ell \ar[rrrrr]^p \ar[d, \text{pr}_1] & & X \\
\Gamma_{K_1, K_2, \mu} \ar[r, \cong, \pi'_{13}] \ar[d, \pi_1] & D_{I_1} \times X \times X^\ell \ar[rr] & & X \times X^\ell \times X \times X^\ell \ar[r, \text{pr}_1] & X \\
D_{K_1} \ar[r, \cong, \pi'] & D_{I_1} \ar[r] & X \times X^\ell
\end{tikzcd}
\end{array}
$$

(7.31)

where $T := (\Gamma_{I_1, J_1, \mu_1} \times X \times X^\ell) \cap (X \times X^\ell \times \Gamma_{I_2, J_2, \mu_2})$, $\pi_{13}$ and $\pi'_{13}$ are the restrictions of the projection $\text{pr}_{13}$, $\pi_1$ and $\pi'_1$ are the restrictions of the projection $\text{pr}_1$, $p$ is the projection to the third factor, and all the other arrows denote the appropriate closed embeddings. Note that $J_1 \cap J_2 \neq \emptyset$ because of the assumption that $n > \ell$. We have

$$T = \left\{(x, x_1, \ldots, x_\ell, y_1, \ldots, y_{n+\ell}, z, z_1, \ldots, z_\ell) \mid x = x_a = y_b = z_c = z, \ x_d = y_{\mu_1(d)} = z_{\mu(d)} \forall a \in K_1, b \in J_1 \cup J_2, c \in K_2, d \in \bar{K}_1 \right\}.$$
We see that a point in $T$ is determined by its $(x, x_1, \ldots, x_\ell)$ component. Thus, $\pi_{13}$ and $\pi_1$ are isomorphisms. Similarly, $\pi'_1$ is an isomorphism. Furthermore, we have $\text{codim } v = \text{codim } \tilde{v} = (k - j + \ell + 1)d$ where $k := |K_1| = |K_2|$. Let $\pi_2 : \Gamma_{K_1, K_2, \mu} \to X \times X^\ell$ be the restriction of the projection $\text{pr}_2 : X \times X^\ell \times X \times X^\ell \to X \times X^\ell$ to the second factor. By the adjunction formula

$$
\pi_{13} \otimes \omega_\ell \cong \omega_\ell \cong \omega_{\pi'_1 \otimes \tilde{v}} \otimes \tilde{v}^* \omega_{\pi'_1}^{-1} \cong \pi_1^* \omega_\alpha \otimes \pi_2^* \omega_{X \times X^\ell}^{-1} \cong p^* \omega_X^{- (k - j)} \otimes \pi_2^* \omega_{X \times X^\ell}^{-1}
$$

where the last isomorphism is due to the fact that on $\Gamma_{K_1, K_2, \mu}$ the projection to the first factor $X \times X^\ell \times X \times X^\ell \to X$ coincides with the projection $p$ to the third factor. It follows that $\pi_{13} \otimes \pi_{2} \omega_{X \times X^\ell} \cong p^* \omega_X^{- (k - j)}$. Note that $|J_1 \cup J_2| = n + k$ and $|J_1 \cap J_2| = n + i + j - k$. One can check that diagram (7.31) with the two bottom lines removed satisfies the properties of diagram (7.15), i.e. the square consisting of $u$, $v$, $s$, and $r$ is a transversal intersection. Using Corollary 7.3.4 together with (7.16) we get

$$
(\mathcal{O}_{I_2, J_2, \mu_2})^R \ast \mathcal{O}_{I_1, J_1, \mu_1} \cong \text{pr}_{13} \cdot \text{Hom}(\text{pr}_{23}^* \mathcal{O}_{I_2, J_2, \mu_2}, \text{pr}_{12}^* \mathcal{O}_{I_1, J_1, \mu_1} \otimes \text{pr}_{2}^* \omega_{X \times X^\ell}((\ell + 1) \cdot d)
$$

$$
\cong \mathcal{O}_{K_1, K_2, \mu} \otimes p^* ((\wedge^q (\omega_X \otimes \theta_{J_1 \cap J_2}) \otimes \omega_X^{-(n+i-1)})) \cdot d.)
$$

(7.32)

Note that here $(\mathcal{O}_{I_2, J_2, \mu_2})^R \ast \mathcal{O}_{I_1, J_1, \mu_1} = \text{pr}_{13} \cdot \left(\text{pr}_{23}^* (\mathcal{O}_{I_2, J_2, \mu_2})^R \otimes \text{pr}_{2}^* \mathcal{O}_{I_1, J_1, \mu_1}\right)$ is the non-equivariant convolution product, i.e. the functor of invariants is not applied; compare (7.8).

7.4.2 The induced maps

Let $c \in \tilde{I}_1$ with $\mu_1(c) \in J_2$ and set $I'_1 = I_1 \cup \{c\}$, $J'_1 = J_1 \cup \{\mu_1(c)\}$, and $\mu'_1 := \mu_1|_{\tilde{I}_1 \setminus \{c\}}$. The restriction $\mathcal{O}_{I_1, J_1, \mu_1} \to \mathcal{O}_{I'_1, J'_1, \mu'_1}$ induces for $q = 0, \ldots, (n + i + j - k)$ a map

$$
\mathcal{H}^{(n+i-1)d-q}\left((\mathcal{O}_{I_2, J_2, \mu_2})^R \ast \mathcal{O}_{I_1, J_1, \mu_1}\right) \to \mathcal{H}^{(n+i-1)d-q}\left((\mathcal{O}_{I_2, J_2, \mu_2})^R \ast \mathcal{O}_{I'_1, J'_1, \mu'_1}\right)
$$

which corresponds under the isomorphism (7.33) to a map

$$
\mathcal{O}_{K_1, K_2, \mu} \otimes p^* ((\wedge^q (\omega_X \otimes \theta_{J_1 \cap J_2}) \otimes \omega_X^{-(n+i-1)})) \to \mathcal{O}_{K_1, K_2, \mu} \otimes p^* ((\wedge^q (\omega_X \otimes \theta_{J'_1 \cap J'_2}) \otimes \omega_X^{-(n+i-1)}))
$$

By Remarks 7.3.5 and 7.3.7, this map is given by the canonical inclusion $\theta_{J_1 \cap J_2} \to \theta_{J'_1 \cap J'_2} = \theta_{(J_1 \cap J_2) \cup \{\mu_1(c)\}}$. To see this, set $\Delta_f = X \times X^\ell \times \Delta_f \times X \times X^\ell$ for $J \subset [n + \ell]$ and consider the diagram

$$
\begin{array}{c}
\begin{array}{c}
\Delta_f \times X^\ell \times \Gamma_{I_2, J_2, \mu_2} \xrightarrow{\text{id}} X \times X^\ell \times \Gamma_{I_2, J_2, \mu_2} \\
\bigwedge_{\Delta_f} \xrightarrow{s} \bigwedge_{\Gamma_{I_1, J_1, \mu_1}} \times X \times X^\ell \xrightarrow{\pi_2} X \times X^\ell \times X^{n+i+j-k} \times X \times X^\ell.
\end{array}
\end{array}
$$

Similarly, consider $c \in \tilde{I}_2$ with $\mu_2(c) \in J_1$ and set $I'_2 := I_2 \cup \{c\}$, $J'_2 := J_2 \cup \{\mu_2(c)\}$, and $\mu'_2 := \mu_2|_{I'_2 \setminus \{c\}}$. Then for $q = 0, \ldots, (n + i + j - k)$ the restriction $\mathcal{O}_{I_2, J_2, \mu_2} \to \mathcal{O}_{I'_2, J'_2, \mu'_2}$ induces a map

$$
\mathcal{H}^{(k-j)d-q}\left((\mathcal{O}_{I'_2, J'_2, \mu'_2})^R \ast \mathcal{O}_{I_1, J_1, \mu_1}\right) \to \mathcal{H}^{(k-j)d-q}\left((\mathcal{O}_{I_2, J_2, \mu_2})^R \ast \mathcal{O}_{I_1, J_1, \mu_1}\right)
$$

187
which corresponds under the isomorphism (7.32) to a map

\[ \mathcal{O}_{K_1,K_2,\mu} \otimes p^*(\lambda^q(T_X \otimes \mathcal{O}_{J_1 \cap J_2}) \otimes \omega_X^{-(k-j)}) \rightarrow \mathcal{O}_{K_1,K_2,\mu} \otimes p^*(\lambda^q(T_X \otimes \mathcal{O}_{J_1 \cap J_2}) \otimes \omega_X^{-(k-j)}) \]

which is given by the canonical surjection \( \mathcal{O}_{J_1 \cap J_2} = \mathcal{O}_{(J_1 \cup J_2) \cup \{\mu_2(e)\}} \rightarrow \mathcal{O}_{J_1 \cap J_2} \). In particular, the induced map \( \mathcal{H}^{(k-j)d}((\mathcal{O}_{I_1,J_2,J_2}^*) \ast \mathcal{O}_{I_1,J_1,J_1}) \rightarrow \mathcal{H}^{(k-j)d}((\mathcal{O}_{I_2,J_2,J_2}^*) \ast \mathcal{O}_{I_1,J_1,J_1}) \) is given by the identity on \( \mathcal{O}_{K_1,K_2,\mu} \otimes p^*\omega_X^{-(k-j)} \).

### 7.4.3 Computation of the \( \mathcal{P}^R \ast \mathcal{P}^j \)

We make use of the principle explained in Section 7.3.5 to compute the convolution product \( \mathcal{P}^R \ast \mathcal{P}^j = \text{pr}_{13+}(\text{pr}_{23}^* \mathcal{P}^R \otimes \text{pr}_{12}^* \mathcal{P})^{1\times \mathcal{S}_{n+\ell} \times 1} \). The \( \mathcal{S}_\ell \times \mathcal{S}_{n+\ell} \times \mathcal{S}_\ell \)-linearisation of

\[ \text{pr}_{13+}(\text{pr}_{23}^* \mathcal{P}^R \otimes \text{pr}_{12}^* \mathcal{P}) \cong \bigoplus_{\text{Index}(j) \times \text{Index}(i)} \text{pr}_{13+}(\text{pr}_{23}^* \mathcal{P}(I_2,J_2,\mu_2) \otimes \text{pr}_{12}^* \mathcal{P}(I_1,J_1,\mu_1)) \]

induces on the index set \( \text{Index}(j) \times \text{Index}(i) \) the action

\[ \sigma_1 \times \tau \times \sigma_2 \colon (I_1,J_1,\mu_1; I_2,J_2,\mu_2) \mapsto (\sigma_1(I_1), \tau(J_1), \tau \circ \mu_1 \circ \sigma_1^{-1}; \sigma_2(I_2), \tau(J_2), \tau \circ \mu_2 \circ \sigma_2^{-1}) \].

Let \( O(i,j) \) be a set of representatives of the \( 1 \times \mathcal{S}_{n+\ell} \times 1 \)-orbits in \( \text{Index}(i) \times \text{Index}(j) \). One can check that \( O(i,j) \) is in bijection with

\[ \text{Index}(i,j) := \left\{ (I_1,K_1,I_2,K_2,\mu) \mid I_1 \subset K_1 \subset [\ell] \supset K_2 \supset I_2, |I_1| = j, |I_2| = i, |K_1| = |K_2|, \mu \colon K_1 \rightarrow K_2 \text{ bijection} \right\} \]

via the assignment \( (I_1,J_1,\mu_1; I_2,J_2,\mu_2) \mapsto (I_1,K_1,I_2,K_2,\mu) \) where

\[ K_1 = I_1 \cup \mu_1^{-1}(J_2) \quad , \quad K_2 = I_2 \cup \mu_2^{-1}(J_1) \quad , \quad \mu = \mu_2^{-1}|_{I_1 \cup J_2} \circ \mu_1|_{K_1} \].

Furthermore, the \( \mathcal{S}_{n+\ell} \)-stabiliser of \( (I_1,J_1,\mu_1; I_2,J_2,\mu_2) \) is \( \mathcal{S}_{J_1 \cup J_2} \). It follows by (7.20) and (7.32) that the equivariant convolution product \( \mathcal{P}^R \ast \mathcal{P}^j \) is given by

\[ \mathcal{P}^R \ast \mathcal{P}^j \]

\[ \cong \text{pr}_{13+}(\text{pr}_{23}^* \mathcal{P}^R \otimes \text{pr}_{12}^* \mathcal{P})^{1\times \mathcal{S}_{n+\ell} \times 1} \]

\[ \cong \bigoplus_{O(i,j)} \text{pr}_{13+}(\text{pr}_{23}^* \mathcal{P}(I_2,J_2,\mu_2) \otimes \text{pr}_{12}^* \mathcal{P}(I_1,J_1,\mu_1))^{1\times \mathcal{S}_{J_1 \cup J_2} \times 1} \] \hspace{1cm} (7.34)

\[ \cong \bigoplus_{\text{Index}(i,j)} \mathcal{O}_{K_1,K_2,\mu} \otimes \text{a}_{K_1 \setminus I_1} \otimes \text{a}_{K_2 \setminus I_2} \otimes p^*(\Lambda_{n+i+j-k}^* \otimes \omega_X^{-(k-j)}) \otimes [-(k-j)d] \] \hspace{1cm} (7.35)

We denote the direct summands of (7.35) by \( \mathcal{Q}(I_1,K_1,I_2,K_2,\mu) \). Note that the \( \mathcal{S}_\ell \times \mathcal{S}_\ell \)-linearisation of \( \mathcal{P}^R \ast \mathcal{P}^j \) induces on \( \mathcal{Q}(i,j) \cong \text{Index}(i,j) \) the action

\[ \sigma_1 \times \sigma_2 \colon (I_1,K_1,I_2,K_2,\mu) \mapsto (\sigma_1(I_1), \sigma_1(K_1), \sigma_2(I_2), \sigma_2(K_2), \sigma_2 \circ \mu \circ \sigma_1^{-1}) \].

The \( \mathcal{S}_\ell \times \mathcal{S}_\ell \)-stabiliser of \( (I_1,K_1,I_2,K_2,\mu) \) is \( \mathcal{S}_{K_1 \setminus I_1} \times \mathcal{S}_{K_1,\mu} \times \mathcal{S}_{K_2 \setminus I_2} \times \mathcal{S}_{K_2 \setminus I_2} \). With this notation we indicate the subgroup of \( \mathcal{S}_\ell \times \mathcal{S}_\ell \) given by

\[ \{ (\sigma_1,\sigma_2) \mid \sigma_1(I_1) = I_1, \sigma_1(K_1) = K_1, \sigma_2(I_2) = I_2, \sigma_2(K_2) = K_2, (\sigma_2 \circ \mu)|_{K_1} = (\mu \circ \sigma_1)|_{K_1} \} \].

188
Furthermore, the orbits of the $\mathcal{G}_\ell \times \mathcal{G}_\ell$-action on $\text{Index}(i,j)$ are given by

$$\text{Index}(i,j)_k := \{|K_1| = |K_2| = k\} \subset \text{Index}(i,j) \quad \text{for} \quad k = \max\{i,j\}, \ldots, \ell.$$  

A representative of the orbit $\text{Index}(i,j)_k$ is $(|j|, |k|, [i], [k], e)$ where $e = \text{id}_{[k+1,\ell]}$. We get

$$\mathcal{P}^R \ast \mathcal{P}^j \cong \bigoplus_{k = \max\{i,j\}}^\ell \text{Inf}_{\mathcal{E}_i \times \mathcal{E}_i}^\mathcal{E}_{\mathcal{E}_{k-j} \times \mathcal{E}_{k-e} \times \mathcal{E}_i \times \mathcal{E}_{k-i}} \mathcal{Q}([j], [k], [i], [k], e). \quad (7.36)$$

We denote the direct summands of (7.36) by

$$\mathcal{P}(i,j)_k := \text{Inf}_{\mathcal{E}_i \times \mathcal{E}_i}^\mathcal{E}_{\mathcal{E}_{k-j} \times \mathcal{E}_{k-e} \times \mathcal{E}_i \times \mathcal{E}_{k-i}} \mathcal{Q}([j], [k], [i], [k], e).$$

### 7.4.4 Spectral sequences

For $j = 0, \ldots, \ell$ there is the spectral sequence $\mathcal{E}(j)$ associated to the complex $\text{pr}^*_{13} \mathcal{P}$ and the functor $\text{Hom}(\mathcal{F}, \text{pr}^*_{13} \mathcal{P}^j)$ given by

$$\mathcal{E}(j)^{p,q}_1 = \mathcal{E}(\mathcal{F}(\text{pr}^*_{13} \mathcal{P}^p, \text{pr}^*_{13} \mathcal{P}^j) \implies \mathcal{E}(j)^{p+q}_1 = \mathcal{E}(\mathcal{F}^p(\text{pr}^*_{13} \mathcal{P}, \text{pr}^*_{13} \mathcal{P}^j);$$

see e.g. [Huy06, Remark 2.67]. By Section 7.4.1, every term of this spectral sequence is finitely supported over $X \times X^\ell \times X \times X^\ell$ hence $\text{pr}_{13}$-acyclic. Since the functors $(\_ \otimes \text{pr}^*_{13})_{\mathcal{G}_1 \times \mathcal{G}_\ell}$ and $(\_ \otimes \text{pr}^*_{13})_{\mathcal{G}_1 \times \mathcal{G}_\ell}$ are exact, we can apply the functor $\text{pr}_{13}(\_ \otimes \mathcal{G}_{n+\ell} \times \mathcal{G}_{n+\ell})$ to every level of the spectral sequence $\mathcal{E}(j)$ to get a spectral sequence with values in $\text{Coh}_{\mathcal{E}_i \times \mathcal{E}_i}(X \times X^\ell \times X \times X^\ell)$. Shifting this spectral sequence by $(\ell + 1)d$ in the $q$-direction we get the spectral sequence $E(j)$ with the property

$$E(j)^{p,q}_1 = \mathcal{H}^q((\mathcal{P}^R \ast \mathcal{P}^j) \implies E(j)^{p+q}_1 = \mathcal{H}^{p+q}(\mathcal{P}^R \ast \mathcal{P}^j).$$

Similarly, we get a spectral sequence

$$E(j)^{p,q}_1 = \mathcal{H}^q(\mathcal{P}^R \ast \mathcal{P}^j) \implies E^{p+q}_1 = \mathcal{H}^{p+q}(\mathcal{P}^R \ast \mathcal{P}). \quad (7.37)$$

### 7.4.5 Long exact sequences

For $k \geq 1$ there is the long exact sequence of $\mathcal{G}_k$-representations

$$0 \rightarrow \mathbb{C} \rightarrow \ldots \rightarrow \text{Inf}_{\mathcal{E}_i \times \mathcal{E}_i}^\mathcal{E}_{a_i} \rightarrow \text{Inf}_{\mathcal{E}_i \times \mathcal{E}_i}^\mathcal{E}_{a_{i+1}} \rightarrow \ldots \rightarrow a_k \rightarrow 0$$

which we denote by $\mathcal{C}^k$. We consider $\mathcal{C}^k$ as a complex in degrees $[0, k]$. The terms are

$$\mathcal{C}^k = \text{Inf}_{\mathcal{E}_i \times \mathcal{E}_i}^\mathcal{E}_{a_i} \cong \bigoplus_{I \subset [k], |I| = i} a_I.$$  

Under this identification the differential $d^i$ is determined by its components $a_I \rightarrow a_J$ which are given by $\varepsilon_{I,b} = (-1)^{|\{u \in I | u < b\}|}$ if $J = I \cup \{b\}$ and which are zero if $I \not\subset J$. That the sequence is exact can be checked either by hand or by considering it as a special case of a Čech complex. We also set $\mathcal{C}^k := \mathcal{C}^k \otimes a_k$. Then $\mathcal{C}^k$ is the exact complex

$$0 \rightarrow a_k \rightarrow \ldots \rightarrow \text{Inf}_{\mathcal{E}_i \times \mathcal{E}_i}^\mathcal{E}_{a_{k-i}} \rightarrow \text{Inf}_{\mathcal{E}_i \times \mathcal{E}_i}^\mathcal{E}_{a_{k-i-1}} \rightarrow \ldots \rightarrow \mathbb{C} \rightarrow 0.$$
Let $M$ be a variety on which we consider $\mathcal{G}_k$ to act trivially. For $E \in \text{Coh}(M)$ we set $\hat{C}^*_k(E) := E \otimes_{\mathcal{G}_k} \hat{C}^*_k$. This is an exact complex in $\text{Coh}_{\mathcal{G}_k}(M)$ given by

$$E \rightarrow \cdots \rightarrow \text{Inf}_{\mathcal{G}_k}^{E_k}(E \otimes a_i) \xrightarrow{d^i(E)} \text{Inf}_{\mathcal{G}_k}^{E_{i+1}}(E \otimes a_{i+1}) \rightarrow \cdots \rightarrow E \otimes a_k.$$  

There also is the exact complex $\hat{C}^*_k(E) := E \otimes_{\mathcal{G}_k} \hat{C}^*_k \cong \hat{C}^*_k(E) \otimes a_k$.

**Lemma 7.4.1.** Let $E \in \text{Coh}(M)$ be simple, i.e. $\text{Hom}(E, E) = \mathbb{C}$. Then

$$\text{Hom}_{\mathcal{G}_k}(\hat{C}_k^i(E), \hat{C}_k^{i+1}(E)) \cong \mathbb{C} \cong \text{Hom}_{\mathcal{G}_k}(\hat{C}_k^i(E), \hat{C}_k^{i+1}(E)).$$

**Proof.** By the adjunction $\text{Inf} \dashv \text{Res}$ we have

$$\text{Hom}_{\mathcal{G}_k}(\hat{C}_k^i(E), \hat{C}_k^{i+1}(E)) \cong [\bigoplus_{|I| = i+1} \text{Hom}(E \otimes a_{[i]}, E \otimes a_I)]_{\mathcal{G}_k \times \mathcal{G}_k} \cong [\bigoplus_{|I| = i+1} \text{Hom}(E, E) \otimes a_{[i]} \otimes a_I \otimes a_{[i]}]_{\mathcal{G}_k \times \mathcal{G}_k}.$$

For $[i] \not\subset I$ we have $|I \setminus [i]| \geq 2$ and hence $(\text{Hom}(E, E) \otimes a_{[i]} \otimes a_{I \setminus [i]})_{\mathcal{G}_k \setminus [i]} = 0$. It follows by Section 7.3.5 that

$$\text{Hom}_{\mathcal{G}_k}(\hat{C}_k^i(E), \hat{C}_k^{i+1}(E)) \cong \text{Hom}(E \otimes a_{[i]}, E \otimes a_{[i+1]} \otimes a_{[i+2,k]} = \mathbb{C}.$$  

The second assertion follows from $\hat{C}_k^i(E) \cong \hat{C}_k^i(E) \otimes a_k$. \qed

**Corollary 7.4.2.** Let $E \in \text{Coh}(M)$ be simple. Then, up to isomorphism, every non-zero $\mathcal{G}_k$-equivariant morphism $\hat{C}_k^i(E) \rightarrow \hat{C}_k^{i+1}(E)$ equals $d^i(E)$ and every non-zero $\mathcal{G}_k$-equivariant morphism $\hat{C}_k(E) \rightarrow \hat{C}_k^{i+1}(E)$ equals $d^i(E)$.

**Convention 7.4.3.** We also define $\hat{C}_0^* = \mathbb{C}[0] = \hat{C}_0^*$ to be the one-term complex with $\mathbb{C}$ in degree zero. Obviously, the complexes $\hat{C}_0^*$ and $\hat{C}_0^*$ are not exact in contrast to the case $k \geq 1$ described above.

### 7.4.6 The curve case: induced maps

We will need the following easy fact in the following.

**Lemma 7.4.4.** Let $\iota : Z \rightarrow M$ be a closed embedding of an irreducible subvarieties. Then we have $\text{Hom}_M(E, \iota_* L) = 0$ for all $L \in \text{Pic}(Z)$ and $E \in \text{Coh}(M)$ with $\text{supp} E \not\subset Z$.

Let $X = C$ be a smooth curve. By Lemma 7.3.8 we have $\Lambda_m^* = \mathcal{O}_C[0]$. Using the results of Section 7.4.3 we get for $k = \max\{i, j\}, \ldots, l$ isomorphisms

$$P(i, j)_k \cong \text{Inf}_{\mathcal{G}_k}^{E_k \times E_k} \big(\mathcal{O}_{[i], [k], c} \otimes a_{[i+1,k]} \otimes a_{[i+1,k]} \otimes p^* \omega_C^{(k-j)}[-(k-j)]\big) \cong \text{Inf}_{\mathcal{G}_k}^{E_k \times E_k} \big(\mathcal{O}_{[i], [k], c} \otimes p^* \omega_C^{(k-j)}[-(k-j)] \otimes a_{[i+1,k]}[-(k-j)]\big).$$

The second isomorphism is due to the general fact that for subgroups $V \subset U \subset G$ of a finite group $G$ there is an isomorphism of functors $\text{Inf}_V^G \cong \text{Inf}_U^G \circ \text{Inf}_V^U$. 

190
Lemma 7.4.5. Let $\max\{i, j\} \leq k \leq \ell$. The component
\[
\mathcal{H}^{k-j}(\mathcal{P}^i \star \mathcal{P}^j) \cong \mathcal{H}^{k-j}(\mathcal{P}(i, j)_{k}) \to \mathcal{H}^{k-j}(\mathcal{P}(i - 1, j)_{k}) \cong \mathcal{H}^{k-j}(\mathcal{P}^{(i-1)R} \star \mathcal{P}^j)
\]
of the morphism $\mathcal{P}^i \star \mathcal{P}^j \to \mathcal{P}^{(i-1)R} \star \mathcal{P}^j$ that is induced by the differential $\mathcal{P}^{i-1} \to \mathcal{P}^i$ is given under the isomorphism (7.38) by $\liminf_{\mathcal{E}_j \times \mathcal{E}_{k-j} \times \mathcal{E}_{\ell-k} \times \mathcal{E}_k} \mathcal{H}^{k-j}((\mathcal{O}[k][k], e) \otimes p^* \omega^{(k-j)}_{\mathcal{C}})$.

Proof. Note that
\[
\mathcal{C}^{k-j}_{k}(\mathcal{O}[k][k], e) \otimes p^* \omega^{(k-j)}_{\mathcal{C}} \cong \bigoplus_{I_2 \subseteq [k], |I_2| = i} \mathcal{Q}(j), [k], I_2, [k], e).
\]
By the previous lemma, all the components $\mathcal{Q}(j), [k], I_2, [k], e) \to \mathcal{Q}(I_1', K_1', I_2', K_2', \mu)$ of the induced map $\mathcal{P}(i, j)_{k} \to \mathcal{P}(i - 1, j)_{k}$ are zero unless $K_1' = K_2' = [k]$ and $\mu = e$. They are also zero for $I_1' \neq [j]$ since $\mathcal{P}^{iR} \star \mathcal{P}^j \to \mathcal{P}^{(i-1)R} \star \mathcal{P}^j$ is given by the identity on the factor $\mathcal{P}^j$ and $\mathcal{Q}(I_1, K_1, I_2, K_2, \mu)$ arises as $\text{pr}_{13*}(\text{pr}_{23*}(\mathcal{P}(I_2, J_2, J_2)_{R} \otimes \text{pr}_{12} \mathcal{P}(I_1, J_1, \mu_1)))^{1 \times \mathcal{E}_j \times \mathcal{E}_k \times \mathcal{E}_{k} \times \mathcal{E}_k}$. Thus, by Corollary 7.4.2 it is sufficient to show that the component
\[
\mathcal{H}^{k-j}(\mathcal{P}(i, j)_{k}) \to \mathcal{H}^{k-j}(\mathcal{P}(i - 1, j)_{k})
\]
of $\mathcal{H}^{k-j}(\mathcal{P}(i, j)_{k}) \to \mathcal{H}^{k-j}(\mathcal{P}(i - 1, j)_{k})$ is non-zero. By (7.34) and (7.35) we have
\[
\mathcal{Q}(j), [k], [i], [k], e) \cong [\mathcal{P}([i], [n + s], e)^{R} \star \mathcal{P}(j), J_1, e)]^{S_{n+i+j-k}}.
\]
where a possible choice of $J_1$ is $J_1 = [n + i + j - k] \cup [n + i + 1, n + k]$. In degree $k - j$ the $S_{n+i+j-k}$-action on $\mathcal{P}([i], [n + s], e)^{R} \star \mathcal{P}(j), J_1, e)$ is trivial since given by the representation $\wedge_0 S_{n+i+j-k}$; see (7.32). Hence,
\[
\mathcal{H}^{k-j}(\mathcal{Q}(j), [k], [i], [k], e) \cong \mathcal{H}^{k-j}(\mathcal{P}([i], [n + s], e)^{R} \star \mathcal{P}(j), J_1, e)) \quad (7.39).
\]
Analogously, we get
\[
\mathcal{H}^{k-j}(\mathcal{Q}(j), [k], [i - 1], [k], e) \cong \mathcal{H}^{k-j}(\mathcal{P}([i - 1], [2, n + i], e)^{R} \star \mathcal{P}(j), J_1, e)) \quad (7.40).
\]
Under (7.39) and (7.40), $\mathcal{H}^{k-j}(\mathcal{Q}(j), [k], [i], [k], e) \to \mathcal{H}^{k-j}(\mathcal{Q}(j), [k], [i - 1], [k], e)$ corresponds to the map
\[
\mathcal{H}^{k-j}(\mathcal{P}([i], [n + i], e)^{R} \star \mathcal{P}(j), J_1, e)) \to \mathcal{H}^{k-j}(\mathcal{P}([i - 1], [2, n + i], e)^{R} \star \mathcal{P}([j, J_1, e))
\]
induced by the restriction $\mathcal{O}_{i-1,[2,n+i],e} \to \mathcal{O}_{i,[n+i],e}$. As pointed out at the end of Section 7.4.2, it is an isomorphism. \qed
7.4.7 The curve case: fully faithfulness

**Proposition 7.4.6.** For $X = C$ a curve we have $\mathcal{P}^R \ast \mathcal{P} \cong \inf_{\mathcal{E} \times \mathcal{F}} \mathcal{O}_{\Delta \times C^{-\ell}}$.

**Proof.** Consider the spectral sequences $E(j)^{p,q} = H^q((\mathcal{P}^{-p})^R \ast \mathcal{P}^j) \implies H^{p+q}(\mathcal{P}^R \ast \mathcal{P}^j)$; see Section 7.4.4. By (7.38) and the previous lemma, the $(k - j)$-th row of $E(j)^1$ is for $k = j, \ldots, \ell$ given by the complex $\inf_{\mathcal{E} \times \mathcal{F}} \mathcal{E}_{j} \times \mathcal{E}_{k-j} \times \mathcal{E}_{k-j-\ell} \times \mathcal{E}_{k} \times \mathcal{O}_{k}[i,j,e] \otimes p^* \omega_X^{-(k-j)} \otimes a_{[j+1,k]}$ shifted into degrees $[-k,0]$. The inflation functor is exact. Thus, all the rows of the spectral sequences are exact with one exception: The zero row of $E(0)^1$ is given by the single non-zero object $E(0)^{0,0} = H^0(\mathcal{P}(0,0)_1)$; see Convention 7.4.3. It follows that $\mathcal{P}^R \ast \mathcal{P}^j = 0$ for $j \geq 1$ and $\mathcal{P}^R \ast \mathcal{P}^0 \cong H^0(\mathcal{P}(0,0)_1)$. Now by the spectral sequence (7.37) or, alternatively, by the fact that $\mathcal{P}^R \ast \mathcal{P}$ is a left convolution of $\mathcal{P}^R \ast \mathcal{P}^0 \to \mathcal{P}^R \ast \mathcal{P}^1 \to \cdots \to \mathcal{P}^R \ast \mathcal{P}^\ell$ it follows that $\mathcal{P}^R \ast \mathcal{P} \cong H^0(\mathcal{P}(0,0)_1) \cong \inf_{\mathcal{E} \times \mathcal{F}} \mathcal{O}_{\Delta \times C^{-\ell}}$. \(\square\)

**Proof of Proposition A(i).** The identity functor $\mathcal{D}^b_{\mathcal{E} \times \mathcal{F}}(C \times C^\ell) \to \mathcal{D}^b_{\mathcal{E} \times \mathcal{F}}(C \times C^\ell)$ equals the equivariant FM transform with kernel $\inf_{\mathcal{E} \times \mathcal{F}} \mathcal{O}_{\Delta \times C^{-\ell}}$; see Section 7.2.1. Note that we have $\mathcal{E}_{\ell} \Delta = \mathcal{E}_{\ell} \mathcal{E}_{\ell} \subset \mathcal{E}_{\ell} \times \mathcal{E}_{\ell}$. Thus, Proposition A(i) follows by Proposition 7.4.6. \(\square\)

7.4.8 The curve case: orthogonality

In this section we will outline the proof of Proposition A(ii). Lemma 7.4.7 and 7.4.8 state formulae for the convolution products $\mathcal{P}^R_{\ell,n} \ast \mathcal{P}^j_{\ell,n}$ for $n + \ell = n' + \ell'$ and the induced maps between them. The proofs, which are analogous to the computations of Sections 7.4.1, 7.4.2, 7.4.3, and 7.4.6, are left to the reader. The author decided to carry out the computations of $\mathcal{P}^R_{\ell',n'} \ast \mathcal{P}^j_{\ell,n}$ in the Sections 7.4.1 and 7.4.3 only in the case $(\ell,n) = (\ell',n')$ in order to avoid the heavier notation. This case is sufficient for the proof of Proposition A(ii) and Theorem C. In particular, the reader mainly interested in Theorem C may skip the rest of the current subsection.

Let now $\ell, n, \ell', n' \in \mathbb{Z}$ be integers such that $n > \max\{1, \ell\}$, $n' > \max\{1, \ell'\}$, and $n + \ell = n' + \ell'$. For $I_1 \subset K_1 \subset [\ell]$, $I_2 \subset K_2 \subset [\ell']$, and $\mu : K_1 \to K_2$ a bijection we consider the subvariety $\Gamma_{K_1,K_2,\mu} \subset X \times X^{\ell} \times X \times X^{\ell'}$ given by

$$\left\{(x,x_1,\ldots,x_\ell,z,z_1,\ldots,z_{\ell'}) \mid x = x_a = z_c \forall a \in K_1, b \in K_2, x_c = z_{\mu(c)} \forall c \in \bar{K}_1\right\}$$

and set $\mathcal{O}_{\Gamma_{K_1,K_2,\mu}} := \mathcal{O}_{\Gamma_{K_1,K_2,\mu}}$ as well as

$$\mathcal{Q}(I_1, K_1, I_2, K_2, \mu) := \mathcal{O}_{\Gamma_{K_1,K_2,\mu}} \otimes a_{K_1 \setminus I_1} \otimes a_{K_2 \setminus I_2} \otimes p^* \left(\Lambda_{n'+i+j-k}^{n'+i+j-k} \otimes \omega_X^{-(k-j)}\right) \mid -(k-j)d$$

where $j := |I_1|$. Again, $p : X \times X^{\ell} \times X \times X^{\ell'} \to X$ denotes the projection to the third factor. For $0 \leq i \leq \ell'$, $0 \leq j \leq \ell$, and $\max\{n'-n+i, j\} \leq k \leq \ell$ we set

$$\mathcal{P}(i, \ell', j, \ell)_k := \inf_{\mathcal{E} \times \mathcal{F}} \mathcal{E}_{j} \times \mathcal{E}_{k-j-\ell} \times \mathcal{E}_{k-j} \times \mathcal{E}_{k+n-n'-i} \mathcal{Q}(\mathcal{E}_k, [k], [i], [k+n-n'], c) \mathcal{O}_{\mathcal{E}_k} \mathcal{O}_{\mathcal{O}_{\Delta \times C^{-\ell}}} .$$

**Lemma 7.4.7.** $\mathcal{P}^R_{\ell,n} \ast \mathcal{P}^j_{\ell,n} \cong \bigoplus_{k=\max\{n'-n+i, j\}}^{\ell} \mathcal{P}(i, \ell', j, \ell)_k$.

**Proof.** This follows from computations analogous to those of Sections 7.4.1 and 7.4.3. \(\square\)
Lemma 7.4.8. Let $X = C$ be a curve and $\ell' > \ell$. Then for $0 \leq j \leq k \leq \ell$ we have the vanishing $\mathcal{H}^{k-j}(\mathcal{P}^i_{\ell,n'} \star \mathcal{P}^j_{\ell,n}) = 0$ for $i > k + n - n'$ and the sequence

$$0 \to \mathcal{H}^{k-j}(\mathcal{P}^{(k+n-n')R}_{\ell,n'} \star \mathcal{P}^j_{\ell,n}) \to \cdots \to \mathcal{H}^{k-j}(\mathcal{P}^{0R}_{\ell,n'} \star \mathcal{P}^j_{\ell,n}) \to 0,$$

whose differentials are induced by the differentials $\mathcal{P}^{i-1} \to \mathcal{P}^i$, is isomorphic to

$$\liminf_{\mathfrak{S}_\ell \times \mathfrak{S}_\ell} \mathcal{C}^\bullet_{k+n-n'}(\mathcal{O}_{[k],[k+n-n'],e} \otimes P^i \omega_{\mathcal{C}}^{-k-j}) \otimes a_{[j+1,k]}.$$

In particular, it is an exact sequence.

Proof. This follows from computations analogous to those of Sections 7.4.2 and 7.4.6.

Proposition 7.4.9. Let $X = C$ be a curve and $\ell' > \ell$. Then $\mathcal{P}^R_{\ell',n'} \star \mathcal{P}_{\ell,n} = 0$.

Proof. This follows from Lemma 7.4.8 together with spectral sequences analogous to those of Section 7.4.4.

7.4.9 The surface case: induced maps

Let $X$ be a smooth surface and still $n > \max\{\ell, 1\}$. We set

$$\tilde{S}_X := (\_ \otimes P^i \omega_X[2]) \in \text{Aut}(D^b_{\mathfrak{S}_\ell \times \mathfrak{S}_\ell}(X \times X^\ell \times X \times X^\ell))$$

and $\tilde{S}^{-[a,b]}_X := \tilde{S}^{-a}_X \oplus \tilde{S}^{-[a+1]}_X \oplus \cdots \oplus \tilde{S}^{-b}_X$ for $a \leq b$ two integers. By Lemma 7.3.8 we have $P^i \Lambda^*_m(X) = \tilde{S}^{-[m-1]}_X(\mathcal{O}_{X \times X^\ell \times X \times X^\ell})$. Hence, for $k = \max\{i,j\}, \ldots, \ell$ we get

$$\mathcal{P}(i,j,k) \cong \tilde{S}^{-[k-i,n+1-\ell]} \liminf_{\mathfrak{S}_\ell \times \mathfrak{S}_\ell} \mathcal{C}^\bullet_{k+n-\ell'}(\mathcal{O}_{[k],[k+1-\ell'],e} \otimes a_{[j+1,k]} \otimes a_{[i+1,k]})$$

Note that the inner Čech complex is taken with respect to $\mathfrak{S}_k$ considered as a subgroup of $\mathfrak{S}_\ell \times \mathfrak{S}_\ell$ by the embedding into the second factor, while the outer Čech complex is taken with respect to $\mathfrak{S}_k$ considered as a subgroup of $\mathfrak{S}_\ell \times \mathfrak{S}_\ell$ by the embedding into the first factor.

Lemma 7.4.10. Let $k, k' \in \max\{i, j\}, \ell$. The components $\mathcal{H}^2(\mathcal{P}(i,j)_k) \to \mathcal{H}^2(\mathcal{P}(i-1,j)_{k'})$ of the morphism $\mathcal{H}^2(\mathcal{P}^R \star \mathcal{P}_j) \to \mathcal{H}^2(\mathcal{P}^R \star \mathcal{P}_j \star \mathcal{P}_j)$ which is induced by the differential $\mathcal{P}^{i-1} \to \mathcal{P}^i$ are zero for $k \neq k'$. Furthermore, for $k - j \leq r \leq n + i - 2$ the component

$$\mathcal{H}^{2r}(\mathcal{P}(i,j)_k) \to \mathcal{H}^{2r}(\mathcal{P}(i-1,j)_k)$$

is given by $\liminf_{\mathfrak{S}_\ell \times \mathfrak{S}_\ell} \mathcal{C}^\bullet_{k+n-\ell'}(\mathcal{O}_{[k],[k+1-\ell'],e} \otimes P^i \omega_{\mathcal{C}}^r)$.

Proof. The components $\mathcal{P}(I'_2, J'_2, \mu'_2) \to \mathcal{P}(I_2, J_2, \mu_2)$ of $\mathcal{P}^{i-1} \to \mathcal{P}^i$ are non-zero only if $I'_2 \subset I_2$. Thus, following the computations of Section 7.4.3, the only components $\mathcal{P}(i,j)_k \to \mathcal{P}(i-1,j)_k$ of $\mathcal{P}^R \star \mathcal{P}_j \to \mathcal{P}(i-1)R \star \mathcal{P}_j$ which are possibly non-zero are those with $k = k'$ or $k-1 = k'$. But $\mathcal{H}^2(\mathcal{P}(i,j)_k) \to \mathcal{H}^2(\mathcal{P}(i-1,j)_{k-1})$ is zero by Lemma 7.4.4. For the proof of the second assertion it is, as in the curve case, sufficient to show that

$$\mathcal{H}^{2r}(\mathcal{Q}([j],[k],[i],[k],e)) \to \mathcal{H}^{2r}(\mathcal{Q}([j],[k],[i-1],[k],e))$$

193
is non-zero; see Corollary 7.4.2. By (7.34) and (7.35) we have
\[ H^2r\left(Q([j],[k],[i],[k],e)\right) \cong H^2r\left(P([i],[n+i],e)^R \ast P([j],J_1,e)\right)^{\mathcal{E}_{n+i+j-k}}. \]
where a possible choice of \(J_1\) is \(J_1 = [n+i+j-k] \cup [n+i+1,n+k]\). Also,
\[ H^2r\left(Q([j],[k],[i-1],[k],e)\right) \cong H^2r\left(P([i-1],[2,n+i],e)^R \ast P([j],J_1,e)\right)^{\mathcal{E}_{[2,n+i+j-k]}} \]
\[ \cong \bigoplus_{a \in [n+i+j-k]} H^2r\left(P([i-1],[n+i] \setminus \{a\},e)^R \ast P([j],J_1,e)\right)^{\mathcal{E}_{[n+i+j-k]}}. \]
The second isomorphism is due to Section 7.3.5. As explained in Section 7.4.2, under the isomorphism (7.32), the components of the induced map
\[ H^2r\left(P([i],[n+i],e)^R \ast P([j],J_1,e)\right) \rightarrow \bigoplus_{a \in [n+i+j-k]} H^2r\left(P([i-1],[n+i] \setminus \{a\},e)^R \ast P([j],J_1,e)\right) \]
are given by the canonical surjections \(\varrho_{[n+i+j-k]} \rightarrow \varrho_{[n+i+j-k] \setminus \{a\}}\). It follows by [Sca09, Lemma B.6(3)] that (7.42) induces an isomorphism on the \(\mathcal{E}_{[n+i+j-k]}\)-invariants. \(\square\)

**Lemma 7.4.11.** Let \(k,k' \in [\max\{i,j+1\},\ell]\). The components \(H^q(P(i,j)k) \rightarrow H^q(P(i,j+1)k')\) of the morphism \(H^q(P^{jR} \ast P^j) \rightarrow H^q(P^{j'}R \ast P^{j+1})\) which is induced by the differential \(P^j \rightarrow P^{j+1}\) are zero for \(k' \notin \{k,k+1\}\). For \(k-r \leq j \leq n+i-1\) the component \(H^2r(P(i,j)k) \rightarrow H^2r(P(i,j+1)k)\) is given by \(\text{ln}_{r}^r \mathcal{E}_k \times \mathcal{E}_k \times \mathcal{E}_k \to \mathcal{E}_k \times \mathcal{E}_k \times \mathcal{E}_k \mathcal{E}_k (\mathcal{O}^k \mathcal{O}_k \mathcal{O}^{k-1} \mathcal{O}^{k-2})\).

**Proof.** The first assertion follows from the fact that the only non-zero components of \(P^j \rightarrow P^{j+1}\) are of the form \(P(I_1,J_1,\mu_1) \rightarrow P(I_1',J_1',\mu_1')\) with \(I_1 \subset I_1'\).

The proof of the second assertion is also analogous to the proof of the second assertion of Lemma 7.4.10: It is sufficient to show the non-vanishing of
\[ H^2r\left(Q([j],[k],[i],[k],e)\right) \rightarrow H^2r\left(Q([j+1],[k],[i],[k],e)\right). \]
Set \(J_2 := [n+i+j+1-k] \cup [n+j+2,n+k]\). There are isomorphisms
\[ H^2r\left(Q([j+1],[k],[i],[k],e)\right) \cong H^2r\left(P([i],[j],J_2,e)^R \ast P([j+1],[n+j+1],e)\right)^{\mathcal{E}_{[n+i+j+1-k]}}. \]
and
\[ H^2r\left(Q([j],[k],[j],[k],e)\right) \cong \bigoplus_{b \in [n+i+j+1-k]} H^2r\left(P([i],[J_2,e)^R \ast P([j],[n+j+1] \setminus \{b\},e)\right)^{\mathcal{E}_{[n+i+j+1-k]}}. \]
Under the isomorphism (7.33), the components of the induced map
\[ \bigoplus_{b \in [n+i+j+1-k]} H^2r\left(P([i],[J_2,e)^R \ast P([j],[n+j+1] \setminus \{b\},e)\right) \rightarrow H^2r\left(P([i],[J_2,e)^R \ast P([j],[n+j+1],e)\right) \]
\[ (7.43) \]
are given by the canonical injections \(\vartheta_{[n+i+j+1-k]} \cdot \{b\} \to \vartheta_{[n+i+j+1-k]}\). It follows by [Sca09, Lemma B.6 (4)] that (7.43) induces an isomorphism on the \(\mathcal{O}_{[n+i+j+1-k]}\)-invariants. \(\square\)

In fact, the component \(\mathcal{H}^{2r}(\mathcal{P}(i,j)_k) \to \mathcal{H}^{2r}(\mathcal{P}(i,j+1)_{k+1})\) corresponds under (7.41) to the map induced by the restriction \(\mathcal{O}_{[k],[k],e} \to \mathcal{O}_{[k+1],[k+1],e}\). But this will not be relevant for our purposes.

### 7.4.10 The surface case: cohomology

Recall that for \(0 \leq m < k\) there is the stupid truncation \(\sigma^\leq_m \mathcal{C}^\bullet_k\) with

\[
\sigma^\leq_m \mathcal{C}^\bullet_k = (0 \to \mathcal{C}^0_k \to \cdots \to \mathcal{C}^m_k \to 0), \quad \mathcal{H}^\alpha(\sigma^\leq_m \mathcal{C}^\bullet_k) = \begin{cases} \coker d^{m-1} & \text{for } \alpha = m \\ 0 & \text{else} \end{cases}
\]

where for \(m = 0\) we have \(\coker d^{-1} = \mathcal{C}^0_k = \mathbb{C}\). For \(\max\{i,j\} \leq k \leq \ell\) we set

\[
\mathcal{R}(i,j,k) := \mathrm{Isom}_{\mathcal{O}^\mathbb{C}}(\mathcal{C}^i_k((T(i,k)) \to \mathcal{H}^k_i(\sigma^\leq k \mathcal{C}^\bullet_k(\mathcal{O}_{[k],[k],e} \otimes p^* \omega_X^{(n+i-1)}))).
\]

**Lemma 7.4.12.** For \(0 \leq j \leq \ell\) and \(1 \leq i \leq \ell\) we have

\[
E(j)_2^{i,2(n+i-1)} \cong \bigoplus_{k = \max\{i,j\}} \mathcal{R}(i,j,k).
\]

For \(j \geq 1\) these are the only non-vanishing terms on the 2-level of \(E(j)\). For \(j = 0\) there are the additional non-vanishing terms \(E(0)_2^{i,2r} = \mathrm{Isom}_{\mathcal{O}^\mathbb{C}}(\mathcal{C}^i_k((\mathcal{O}_{[k],[k],e} \otimes p^* \omega_X^{(n+i-1)})))\) for \(r = 0, \ldots, n-1\).

**Proof.** The terms \(E(j)_2^{i,2r} = \mathcal{H}^r(\mathcal{P}^{-pR \ast} \mathcal{P}j)\) are described by (7.36) together with (7.41). We see that the only non-vanishing rows on the 1-level of \(E(j)\) have \(q = 2r\) where \(r = 0, \ldots, n-1\). By Lemma 7.4.10 for \(i \geq 1\) the row \(q = 2(n+i-1)\) is the complex

\[
\sigma^\leq_{-i}\left( \bigoplus_{k = \max\{i,j\}}^\ell \mathrm{Isom}_{\mathcal{O}^\mathbb{C}}(\mathcal{C}^i_k((\mathcal{O}_{[k],[k],e} \otimes p^* \omega_X^{(n+i-1)})[k]) \right)
\]

\[
\cong \bigoplus_{k = \max\{i,j\}}^\ell \sigma^\leq_{-i}\left( \mathrm{Isom}_{\mathcal{O}^\mathbb{C}}(\mathcal{C}^i_k((\mathcal{O}_{[k],[k],e} \otimes p^* \omega_X^{(n+i-1)})[k]) \right).
\]

Since the functor \(\mathrm{Isom}_{\mathcal{O}^\mathbb{C}}(\mathcal{C}^i_k((\mathcal{O}_{[k],[k],e} \otimes p^* \omega_X^{(n+i-1)})))\) is exact, it follows that the cohomology of the row \(q = 2(n+i-1)\) is concentrated in degree \(-i\) and equal to \(\bigoplus_{k = \max\{i,j\}}^\ell \mathcal{R}(i,j,k)\). This proofs the first assertion. For \(r = 0, \ldots, n-1\) the row \(q = 2r\) of \(E(j)_1\) is given by

\[
\bigoplus_{k = j}^\ell \mathrm{Isom}_{\mathcal{O}^\mathbb{C}}(\mathcal{C}^i_k((\mathcal{O}_{[0],[0],e} \otimes p^* \omega_X^{(n+i-1)}))[k])\]

Thus, it is an exact complex with one exception: In the case \(j = 0\) the one-term complex \(\mathrm{Isom}_{\mathcal{O}^\mathbb{C}}(\mathcal{C}^i_k((\mathcal{O}_{[0],[0],e} \otimes p^* \omega_X^{(n+i-1)}))[0]) \cong \mathrm{Isom}_{\mathcal{O}^\mathbb{C}}(\mathcal{O}_{\Delta_{X \times X \ell}} \otimes p^* \omega_X^{(n+i-1)}[0])\) occurs as a direct summand; compare Convention 7.4.3. \(\square\)
Corollary 7.4.13. For \( j = 1, \ldots, \ell \) we have
\[
\mathcal{H}^q(P^R \ast P^j) = \begin{cases} 
\bigoplus_{k=\max\{i,j\}}^\ell \mathcal{R}(i,j,k) & \text{for } q = 2(n-1) + i, i = 1, \ldots, \ell \\
0 & \text{else}. 
\end{cases}
\]

Furthermore,
\[
\mathcal{H}^q(P^R \ast P^0) = \begin{cases} 
\bigoplus_{k=\max\{i,j\}}^\ell \mathcal{R}(i,0,k) & \text{for } q = 2(n-1) + i, i = 1, \ldots, \ell \\
\inf \otimes E_{t,e} \Delta_{X \times X^t} \otimes p^* \omega_X^{-r} & \text{for } q = 2r, r = 0, \ldots, n-1 \\
0 & \text{else}. 
\end{cases}
\]

Proof. By the positioning of the non-vanishing terms, we see that all the \( E(j) \) degenerate at the 2-level. The result follows since \( E(j)^q = \mathcal{H}^q(P^R \ast P^j) \); see Section 7.4.4.

Lemma 7.4.14. Let \( A \) be an abelian category and \((C^\alpha, d^\alpha)\) be complexes in \( A \) for \( \alpha = 1, \ldots, m \). Let \( C^\bullet \) be a complex with terms \( C^j = C^j_1 \oplus \cdots \oplus C^j_m \) and differentials of the form
\[
d^j = \begin{pmatrix} d^j_1 & 0 & \cdots & 0 \\ \ast & d^j_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \ast & \cdots & \ast & d^j_m \end{pmatrix}
\]
where the stars stand for arbitrary morphisms. Then, if all the \( C^\alpha \) are exact, \( C^\bullet \) is exact too.

Proof. Let \( B^\bullet \) be the complex with terms \( B^j = C^j_1 \oplus \cdots \oplus C^j_{m-1} \) and differentials
\[
d^j_B = \begin{pmatrix} d^j_1 & 0 & \cdots & 0 \\ \ast & d^j_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \ast & \cdots & \ast & d^j_{m-1} \end{pmatrix}
\]

By induction, we can assume that \( B^\bullet \) is an exact complex. There is the short exact sequence of complexes \( 0 \to C^\bullet \to C^\bullet \to B^\bullet \) where the first map is given by the inclusion of the last direct summand and the second map is the projection to the first \( m - 1 \) direct summands. The exactness of \( C^\bullet \) follows from the associated long exact cohomology sequence.

For \( r \in \mathbb{Z} \) we set \( \mathcal{S}^r_X := \mathcal{S}^r_X(\inf \otimes E_{t,e} \Delta_{X \times X^t}) = \inf \otimes E_{t,e} \Delta_{X \times X^t} \otimes p^* \omega_X^{-r}[2r] \). We have \( \mathcal{S}^r_X(\mathcal{S}^1_X)^r \) and
\[
FM(\mathcal{S}^r_X) = \mathcal{S}^r_X = (\mathcal{S}^{r}_{X}) \otimes (\omega_X^r \boxtimes \Delta_{X \times X^t})[2r] : D_{X^t}(X \times X^t) \to D_{X^t}(X \times X^t).
\]

Proposition 7.4.15. \( \mathcal{H}^*(P^R \ast P) = \mathcal{S}_X^*[0,n-1] := \mathcal{S}_X^0 \oplus \mathcal{S}_X^{-1} \oplus \cdots \oplus \mathcal{S}_X^{-(n-1)} \).

Proof. We consider the spectral sequence \( E_1^{p,q} = \mathcal{H}^q(P^R \ast P^p) \Rightarrow \mathcal{H}^{p+q}(P^R \ast P) \); see (7.37). By Corollary 7.4.13 the only non-vanishing rows of \( E_1 \) are
\[
q = 0, 2, \ldots, 2(n-1), 2(n-1) + 1, \ldots, 2(n-1) + \ell.
\]

196
Note that the terms of the row \( q = 2(n - 1) + i \) for \( i = 1, \ldots, \ell \) equal those of the exact complex \( \bigoplus_{k=i}^{\ell} \inf_{E_k} \mathbb{C}(T(i, k)) \). We set \( \mathcal{R}(i, j, k) = 0 \) and \( d^{i}_{k} = 0 \) for \( j > k \). By Lemma 7.4.11 the map \( d_{i}^{j} : E_{i}^{j} = \bigoplus_{k=i}^{\ell} \mathcal{R}(i, j, k) \to E_{i+1}^{j+1} = \bigoplus_{k=i}^{\ell} \mathcal{R}(i, j+1, k) \) is given by

\[
d_{i}^{j} = \begin{pmatrix}
d_{i}^{j} & 0 & \cdots & 0 \\
* & d^{i}_{i+1} & \cdots & 0 \\
0 & \ddots & \ddots & \vdots \\
0 & 0 & * & d^{i}_{\ell}
\end{pmatrix},
\]

\( d^{i}_{k} = \inf_{E_k} \mathbb{C}(T(i, k)) \).

It follows by Lemma 7.4.14 that the row \( q = 2(n - 1) + i \) is exact for all \( i = 1, \ldots, \ell \). For \( r = 0, \ldots, n - 1 \), the row \( q = 2r \) has only one non-vanishing term, namely \( E_{1}^{2r} = \mathcal{S}^{-r} \). In summary, the only non-zero terms on the 2-level are \( E_{2}^{2r} = \mathcal{S}^{-r} \) for \( r = 0, \ldots, n - 1 \) from which the proposition follows.

\[\square\]

### 7.4.11 The surface case: splitting and monad structure

**Lemma 7.4.16.** Let \( A \) be an abelian category with enough injectives and \( A^{\bullet}, B^{\bullet}, C^{\bullet} \in D^{b}(A) \) together with morphisms \( f : A^{\bullet} \to C^{\bullet}, g : B^{\bullet} \to C^{\bullet} \) in \( D^{b}(A) \). Let there be an \( m \in \mathbb{Z} \) such that the cohomology of \( A^{\bullet} \) and \( B^{\bullet} \) is concentrated in degrees smaller than \( m \) and such that \( H^{i}(f) \) as well as \( H^{i}(g) \) are isomorphisms for all \( i < m \). Then \( A^{\bullet} \cong B^{\bullet} \) in \( D^{b}(A) \).

**Proof.** We may assume that \( A^{i} = B^{i} = 0 \) for all \( i \geq m \); see [Huy06, Exercise 2.31]. Choose an injective complex \( I^{\bullet} \) which is quasi-isomorphic to \( C^{\bullet} \). Then \( f \) and \( g \) are represented by morphisms of complexes \( f^{\bullet} : A^{\bullet} \to I^{\bullet} \) and \( g^{\bullet} : B^{\bullet} \to I^{\bullet} \); see [Huy06, Lemma 2.39]. These morphisms factor through the smart truncation \( \tau^{m-1}I^{\bullet} \) and by the hypothesis these factorisations are quasi-isomorphisms. Thus, they are isomorphisms in \( D^{b}(A) \) which proves the assertion.

\[\square\]

**Proposition 7.4.17.** \( \mathcal{P}^{R} \ast \mathcal{P} \cong \mathcal{S}^{-[0,n-1]} \).

**Proof.** This follows by applying the previous lemma to the situation that \( f : \mathcal{P}^{R} \ast \mathcal{P} \to \mathcal{P}^{R} \ast \mathcal{P}^{0} \) is the map induced by the canonical map \( \mathcal{P} \to \mathcal{P}^{0} \) and \( g \) is the composition

\[
\mathcal{S}^{-[0,n-1]} \to \mathcal{P}^{0R} \ast \mathcal{P}^{0} \to \mathcal{P}^{R} \ast \mathcal{P}^{0}
\]

where the first map is the inclusion of the direct summand \( \mathcal{P}(0,0)_{0} \cong \mathcal{S}^{-[0,n-1]} \) under the isomorphism (7.36) and the second map is induced by \( \mathcal{P} \to \mathcal{P}^{0} \).

\[\square\]

**Proof of Theorem C.** By the previous proposition, \( H_{\ell,n} \) fulfils condition (i) of a \( \mathbb{P}^{n-1} \)-functor.

Set \( F_{\ell,n} := \delta_{[n]} \circ M_{\ell,n} \circ \text{triv} : D^{b}_{E_{\ell,n}}(X \times X^{\ell}) \to D^{b}_{\mathcal{S}_{\ell,n} \times \mathcal{S}_{\ell,n}}(X^{n} \times X^{\ell}) \) so that \( H^{0}_{\ell,n} = \inf \circ F_{\ell,n} \); compare (7.11). By (7.18) we have \( F^{0}_{\ell,n} F_{\ell,n} = \mathcal{S}^{-[0,n-1]} \). The unit of adjunction \( \eta : \text{id} \to \text{Res} \circ \inf \) gives a map of monads \( F^{R}_{\ell,n} \eta F_{\ell,n} : F^{R}_{\ell,n} F_{\ell,n} \to H^{0R}_{\ell,n} H^{0}_{\ell,n} \). On the level of the FM kernels it coincides with the inclusion \( \mathcal{S}^{-[0,n-1]} \to \mathcal{P}^{0R} \ast \mathcal{P}^{0} \). Since \( F_{\ell,n} = F_{0,n} \otimes \text{id}_{D^{b}_{E_{\ell,n}}(X^{\ell})} \), the monad multiplication \( \mu(F_{\ell,n}) : F^{R}_{\ell,n} F_{\ell,n} F^{R}_{\ell,n} F_{\ell,n} \to F^{R}_{\ell,n} F_{\ell,n} \) equals \( \mu(F_{0,n}) \otimes \text{id} \). By [6], \( F_{0,n} \) is a \( \mathbb{P}^{n-1} \)-functor. In particular, the monad structure of \( F_{0,n} \) has the right shape which means that the components \( \mathcal{S}^{-k} \circ \mathcal{S}^{-k} \to \mathcal{S}^{-k+1} \) of \( \mu(F_{0,n}) \) are isomorphisms for \( k = 0, \ldots, n - 2 \). Thus, also the components \( \mathcal{S}^{-k} \circ \mathcal{S}^{-k} \to \mathcal{S}^{-k+1} \) of \( \mu(F_{\ell,n}) \) are isomorphisms for \( k = 0, \ldots, n - 2 \).
Equivalently, on the level of the FM kernels the components $\tilde{S}^{-1}_X \circ S^{-k}_X \to S^{-(k+1)}_X$ of the monad multiplication

$$\mathcal{P}(\emptyset, [n], e^R) \star \mathcal{P}(\emptyset, [n], e) \star \mathcal{P}(\emptyset, [n], e^R) \star \mathcal{P}(\emptyset, [n], e) \to \mathcal{P}(\emptyset, [n], e^R) \star \mathcal{P}(\emptyset, [n], e),$$

which we denote again by $\mu(F_{\ell,n})$, are isomorphisms. Let $U := (X \times X^\ell) \setminus (\cup_{0 \neq I \subseteq [\ell]} D_I)$ and $u: U \to X \times X^\ell$ the open embedding. Then $H_{\ell,n} \circ u_* \cong H^0_{\ell,n} \circ u_*$ and $F^{R,\eta}_{\ell,n}$ is an isomorphism over $U \times U$. It follows that the components $\tilde{S}^{-1}_X \circ S^{-k}_X \to S^{-(k+1)}_X$ of $\mu(H_{\ell,n})$ are isomorphisms over the whole $X \times X^\ell \times X \times X^\ell$. This amounts to condition (ii) of a $\mathbb{D}^{n-1}$-functor.

That the $H_{\ell,n}$ satisfy condition (iii) of a $\mathbb{D}^{n-1}$-functor was already shown in Section 7.2.5.

7.4.12 The case of the generalised Kummer stacks

Let $A$ be an abelian variety of dimension $d$. For $\ell, n \in \mathbb{N}$ with $n > \max\{\ell, 1\}$ we set

$$M_{\ell,n} A := \{(a, a_1, \ldots, a_\ell) \mid n \cdot a + a_1 + \cdots + a_\ell = 0) \subset A \times A^\ell,$$

$$N_{n+\ell-1} := \{(b_1, \ldots, b_{n+\ell}) \mid b_1 + \cdots + b_{n+\ell} = 0) \subset A^{n+\ell}.$$

Note that these subvarieties are invariant under the $\mathcal{G}_{\ell}$-action on $A \times A^\ell$ and the $\mathcal{G}_{n+\ell}$-action on $A^{n+\ell}$, respectively. We set $\hat{\Gamma}_{I,J,\mu} := \Gamma_{I,J,\mu} \cap (M_{\ell,n} A \times N_{n+\ell-1} A)$ for $(I, J, \mu) \in \text{Index}(i)$ where $\Gamma_{I,J,\mu} \subset A \times A^\ell \times A^{n+\ell}$ is the subvariety described in Section 7.2.4. Let $\hat{\mathcal{P}}(I, J, \mu) := O_{\hat{\Gamma}_{I,J,\mu}} \otimes a_I$ and

$$\hat{\mathcal{P}}^i := \text{inf}_{\hat{\mathcal{P}}_{\hat{\mathcal{G}}_{\ell} \times \hat{\mathcal{G}}_{n+\ell}}^i} \hat{\mathcal{P}}([i], [n+i], e) = \bigoplus_{\text{Index}(i)} \hat{\mathcal{P}}(I, J, \mu).$$

We set $\hat{H}_{\ell,n} := \text{FM}_A : D^b_{\hat{\mathcal{G}}_{\ell}}(M_{\ell,n} A) \to D^b_{\hat{\mathcal{G}}_{n+\ell}}(N_{n+\ell-1} A)$ where $\hat{\mathcal{P}}$ is the complex

$$\hat{\mathcal{P}} := \hat{\mathcal{P}}_{\ell,n} = (0 \to \hat{\mathcal{P}}^0 \to \cdots \to \hat{\mathcal{P}}^\ell \to 0) \in D^b_{\hat{\mathcal{G}}_{\ell} \times \hat{\mathcal{G}}_{n+\ell}}(M_{\ell,n} A \times N_{n+\ell-1} A).$$

For $I \subset [n+\ell]$ with $|I| = n$, the morphism $\delta_I: A \times A^\ell \to A^{n+\ell}$ restricts to a morphism $\hat{\delta}_I: M_{\ell,n} A \to N_{n+\ell-1} A$. The functor $\hat{H}^0_{\ell,n}$ is given by the composition

$$D^b_{\hat{\mathcal{G}}_{\ell}}(M_{\ell,n} A) \xrightarrow{\text{M}_{\ell,n} \circ \text{triv}} D^b_{\hat{\mathcal{G}}_{n+\ell} \times \hat{\mathcal{G}}_{\ell}}(M_{\ell,n} A) \xrightarrow{\delta_{I,\mu}^* \downarrow} D^b_{\hat{\mathcal{G}}_{n+\ell} \times \hat{\mathcal{G}}_{\ell}}(N_{n+\ell-1} A) \xrightarrow{\text{inf}} D^b_{\hat{\mathcal{G}}_{n+\ell}}(N_{n+\ell-1} A).$$

(7.46)

Remark 7.4.18. Let $\iota: M_{\ell,n} A \to A \times A^\ell$ and $\iota': N_{n+\ell-1} A \to A^{n+\ell}$ denote the closed embeddings. We have $\bar{\mathcal{P}}(I, J, \mu) \not\cong (\iota \times \iota')^* \hat{\mathcal{P}}(I, J, \mu)$ where $(\iota \times \iota')^*$ denotes the derived pull-back (note that the equality holds if we consider the non-derived pull-back instead). The reason is that $\Gamma_{I,J,\mu}$ and $M_{\ell,n} A \times N_{n+\ell-1} A$ do not intersect transversally inside of $A \times A^\ell \times A^{n+\ell}$. Indeed,

$$\text{codim}(M_{\ell,n} A \times N_{n+\ell-1} A \hookrightarrow A \times A^\ell \times A^{n+\ell}) = 4d,$$

$$\text{codim}(\hat{\Gamma}_{I,J,\mu} \hookrightarrow \Gamma_{I,J,\mu}) = 2d.$$
Lemma 7.4.19. We have \((\iota \times \id)_*\mathcal{P}^i \cong (\id \times \iota')^*\mathcal{P}^i\) and \((\id \times \iota')_*\mathcal{P}^i \cong (\iota \times \id)^*\mathcal{P}^i\) for every \(i = 0, \ldots, \ell\). Also, \(((\iota \times \id)_*\mathcal{P}^i \cong (\id \times \iota')^*\mathcal{P}^i\) and \((\id \times \iota')_*\mathcal{P}^i \cong (\iota \times \id)^*\mathcal{P}^i\).

Proof. For every \(0 \leq i \leq \ell\) and \((I, J, \mu) \in \text{Index}(i)\) the two diagrams of closed embeddings

\[
\begin{array}{ccc}
\hat{\Gamma}_{I,J,\mu} & \longrightarrow & \Gamma_{I,J,\mu} \\
\downarrow & & \downarrow \\
A \times A^\ell \times N_{\ell+1} & \longrightarrow & A \times A^\ell \times A^{\ell+\ell} \\
\end{array}
\]

are transversal intersections. It follows from the base change [Kuz06, Corollary 2.27] that

\[(\iota \times \id)_*\mathcal{P}(I, J, \mu) \cong (\id \times \iota')^*\mathcal{P}(I, J, \mu),\quad (\id \times \iota')_*\mathcal{P}(I, J, \mu) \cong (\iota \times \id)^*\mathcal{P}(I, J, \mu)\]

for all \((I, J, \mu) \in \text{Index}(i)\). The result follows from the definition of \(\mathcal{P}^i\) as a direct sum of the \(\mathcal{P}(I, J, \mu)\) and \((7.45)\).

Corollary 7.4.20. The functor \(H_{\ell,n}\) is a restriction of \(H_{\ell,n}\) in the sense that there are isomorphisms \(H_{\ell,n} \circ \iota^* \cong \iota'^* \circ H_{\ell,n}\) and \(\iota'_* \circ H_{\ell,n} \cong H_{\ell,n} \circ \iota_*\).

Proposition 7.4.21. We have

\[\mathcal{H}^*(\hat{\mathcal{P}}^R \star \hat{\mathcal{P}}) \cong \begin{cases} \text{Inf}^{\mathcal{E}_\ell \times \mathcal{E}_\ell_1} \mathcal{O}_{\Delta[0]} & \text{for } A = E \text{ an elliptic curve}, \\ \text{Inf}^{\mathcal{E}_\ell \times \mathcal{E}_\ell_1} \mathcal{O}_{\Delta(\{0\} \oplus [-2] \oplus \cdots \oplus [-2(n-1)]]} & \text{for } A \text{ an abelian surface.} \end{cases}\]

Furthermore, for \(\ell', n' > \ell > \ell'\) and \(n' + \ell' = n + \ell\) we have \(\mathcal{H}^*(\hat{\mathcal{P}}^R_{\ell', n'} \star \hat{\mathcal{P}}_{\ell, n}) = 0\) in the case of an elliptic curve.

Proof. It follows from the previous corollary taking right adjoints that

\[\iota_* \circ \hat{H}^R_{\ell,n} \circ \hat{H}_{\ell,n} \cong H^R_{\ell,n} \circ H_{\ell,n} \circ \iota_*\]

The FM kernel of the left hand side is \((\id \times \iota)_*\mathcal{D}^R_{\ell,n} \cong \mathcal{D}^R_{\ell,n}(M_{\ell,n}A \times A \times A^\ell)\) and the FM kernel of the right hand side is

\[(\iota \times \id)^*(\mathcal{P}^R \star \mathcal{P}) \cong \begin{cases} \text{Inf}^{\mathcal{E}_\ell \times \mathcal{E}_\ell_1} \mathcal{O}_{\Delta[0]} & \text{for } \dim A = 1, \\ \text{Inf}^{\mathcal{E}_\ell \times \mathcal{E}_\ell_1} \mathcal{O}_{\Delta(\{0\} \oplus [-2] \oplus \cdots \oplus [-2(n-1)]]} & \text{for } \dim A = 2. \end{cases}\]

This follows from the Propositions 7.4.6 and 7.4.17 together with the fact that

\[
\begin{array}{ccc}
\Gamma_\ell & \longrightarrow & \Delta_{A \times A^\ell} \\
\downarrow & & \downarrow \\
M_{\ell,n}A \times A \times A^\ell & \longrightarrow & A \times A^\ell \times A \times A^\ell \\
\end{array}
\]

is a transversal intersection. Since \((\id \times \iota)_*\) is an exact functor on the level of coherent sheaves, we have \(\mathcal{H}^*((\id \times \iota)_*(\mathcal{P}^R \star \mathcal{P})) \cong (\id \times \iota)_* \mathcal{H}^*((\mathcal{P}^R \star \mathcal{P}))\). The formulae for \(\mathcal{H}^*((\mathcal{P}^R \star \mathcal{P}))\) follow from the uniqueness of the cohomology of FM kernels; see [CS12, Theorem 1.2]. Analogously, the vanishing of \(\mathcal{H}^*((\mathcal{P}^R_{\ell', n'} \star \mathcal{P}_{\ell, n}))\) in the case of a curve follows from Proposition 7.4.9.
The case of an elliptic curve proves Proposition A'.

Proof of Theorem C'. Let $A$ be an abelian surface. We need to compute one component of the convolution product $\hat{P}^{0R} \ast \hat{P}^0$, namely $[\hat{P}_{13}(\hat{p}_{23}^{\ast} \hat{P}(\emptyset, [n], e)^R \otimes \hat{p}_{12}^{\ast} \hat{P}(\emptyset, [n], e))]^{\mathcal{S}^n}$. There is the diagram

\[
\begin{array}{ccc}
\hat{T} & \stackrel{\hat{\mu}}{\longrightarrow} & M_{\ell,n}^\ast A \\
\downarrow & & \downarrow \hat{\eta} \\
\hat{\Delta}_{\emptyset, [n], e} \times M_{\ell,n}^\ast A & \stackrel{\hat{i}}{\longrightarrow} & M_{\ell,n}^\ast A \times N_{\ell+1} \times M_{\ell,n}^\ast A \\
\downarrow & & \downarrow \hat{\nu} \\
\hat{\Delta}_{\emptyset, [n], e} \times M_{\ell,n}^\ast A & \stackrel{\hat{i}_1}{\longrightarrow} & M_{\ell,n}^\ast A \times M_{\ell,n}^\ast A \\
\end{array}
\] (7.47)

where $\hat{T} = (M_{\ell,n}^\ast A \times \hat{\Delta}_{\emptyset, [n], e}) \cap (\hat{\Delta}_{\emptyset, [n], e} \times M_{\ell,n}^\ast A)$ and $\hat{\Delta}_{[n]} := \hat{\Delta}_{[n]} \cap N_{\ell+1}$. The upper part is a diagram of closed embeddings with the same properties as diagram (7.15). The restriction $\hat{\pi}$ of the projection $\hat{p}_{13}$ is an isomorphism onto the diagonal and codim $\hat{\pi} = \text{codim} \hat{\pi} = 2\ell = \dim M_{\ell,n}^\ast A$. Thus, analogously to Sections 7.4.1 and 7.4.3, one computes

\[ [\hat{p}_{13}(\hat{P}_{23}^{\ast} \hat{P}(\emptyset, [n], e)^R \otimes \hat{p}_{12}^{\ast} \hat{P}(\emptyset, [n], e))]^{\mathcal{S}^n} \cong O_{\Delta_{\emptyset,n}}([0] \oplus [-2] \oplus \cdots \oplus [-2(n-1)]).
\]

Thus, $\hat{P}(0,0) := \inf_{\mathcal{S}_{\ell,n}} O_{\Delta}([0] \oplus [-2] \oplus \cdots \oplus [-2(n-1)])$ occurs as a direct summand of $\hat{P}^{0R} \ast \hat{P}^0$. By the same arguments as in the proof of Proposition 7.4.21 one can deduce from Lemma 7.4.19 that $H^*(\hat{P}^{0R} \ast \hat{P}^0) \cong (i \times i)^* H^*(\mathcal{P}^{0R} \ast \mathcal{P}^0)$ and $H^*(\hat{P}^{0R} \ast \hat{P}^0) \cong (i \times i)^* H^*(\mathcal{P}^{0R} \ast \mathcal{P}^0)$ for all $i, j \in [\ell]$. Here, for once $(i \times i)^*$ denotes the non-derived pull-back. In particular, all the $H^*(\hat{P}^{0R} \ast \hat{P}^0)$ with $(i, j) \neq (0, 0)$ are supported on proper subsets of the components of $\text{supp} \hat{P}(0,0)$. By Lemma 7.4.4, it follows that the composition $\hat{P}(0,0) \to \hat{P}^{0R} \ast \hat{P}^0 \to \hat{P}^R \ast \hat{P}^0$ induces an isomorphism on the cohomology in the degrees $\leq 2(n-1)$. Furthermore, $H^*(\hat{P}^{0R} \ast \hat{P}^0) = 0$ for $j \geq 1$ and $k \leq 2(n-1)$ by Corollary 7.4.13. Hence, also the morphism $\hat{P}^R \ast \hat{P} \to \hat{P}^R \ast \hat{P}^0$ induces an isomorphism on cohomology in the degrees $\leq 2(n-1)$.

Thus, we can apply Lemma 7.4.16 to conclude that $\hat{P}^{0R} \ast \hat{P} \cong \hat{P}(0,0)$. It follows that $H_\ell^{R \ast} \circ \hat{H}_\ell \ast \cong \text{id} [-2] \oplus \cdots \oplus [-2(n-1)]$ which proves condition (i) of a $\mathbb{P}^{n-1}$-functor.

We set $\hat{F}_{\ell,n} := \delta_{[n]} \ast M_{\delta_{[n]}} \circ \text{triv}$ so that $\hat{H}_{\ell,n} \cong \hat{F}_{\ell,n}$; compare (7.46). Since the diagram

\[
\begin{array}{ccc}
M_{\ell,n}^\ast A & \stackrel{\delta_{[n]}}{\longrightarrow} & N_{\ell+1}^\ast A \\
\downarrow \hat{i} & & \downarrow \hat{i}' \\
A \times A^\ell & \stackrel{\delta_{[n]}}{\longrightarrow} & A^{\ell+1} \\
\end{array}
\]

is a transversal intersection, we have $N_{\ell+1}^\ast \cong N_{\delta_{[n]}} \ast M_{\ell,n} \ast A$ as $\mathcal{S}_n$-bundles. Hence,

\[ \hat{F}_{\ell,n} \circ \hat{F}_{\ell,n} \cong (-) \otimes A_n(A) \ast M_{\ell,n} \ast A \cong \text{id} [-2] \oplus \cdots \oplus [-2(n-1)]; \]
see Section 7.3.2. By [6, Lemma 3.3] it follows that \( \tilde{H}_{l,n} \) also fulfills condition (ii) of a \( \mathbb{P}^{n-1} \)-functor with cotwist \([-2]\). Now condition (ii) of a \( \mathbb{P}^{n-1} \)-functor for \( H_{l,n} \) can be deduced the same way as it was done for \( H_{l,n} \) in Section 7.4.11.

Since the \( \mathbb{P} \)-cotwist as well as the Serre functors are just shifts, condition (iii) can be verified by a simple dimension count. \( \square \)

### 7.5 Interpretation of the results

#### 7.5.1 Spherical and \( \mathbb{P} \)-twists

The \( \mathbb{P} \)-twist associated to a \( \mathbb{P}^n \)-functor \( F: D_b^G(M) \to D_b^H(N) \) with \( \mathbb{P} \)-cotwist \( D \) is defined as the double cone

\[
P_F := \text{cone}\left( \text{cone}(FDFR \to FR) \to \text{id} \right).
\]

The map defining the inner cone is given by the composition

\[
FDFR \xrightarrow{FjFR} FRFR \xrightarrow{\epsilon FR-FFR} FR
\]

where \( j \) is the inclusion of the direct summand \( D \). The map defining the outer cone is induced by the counit \( \epsilon: FR \to \text{id} \); for details see [Add16, Section 3.3]. The functor \( P_F: D_b^H(N) \to D_b^H(N) \) is always an autoequivalence; see [Add16, Theorem 3]. On the spanning class \( \text{im} F \cup \ker FR = \text{im} F \cup (\text{im} F)^\perp \) the twist \( P_F \) is given by

\[
P_F \circ F \cong F \circ D^{n+1}[2] , \quad P_F(B) = B \quad \text{for } B \in \ker FR .
\]

For \( S: D_b^G(M) \to D_b^H(N) \) a spherical functor the associated spherical twist is defined as the cone \( T_S := \text{cone}(S \circ S^R \xrightarrow{\epsilon} \text{id}) \) of the counit. It is an autoequivalence with

\[
T_S \circ F \cong F \circ C[1] , \quad T_S(B) = B \quad \text{for } B \in \ker S^R ;
\]

see e.g. [Add16, Section 1]. In the case that \( S \) is split spherical, i.e. a \( \mathbb{P}^1 \)-functor, \( T_S^2 \cong P_S \). Let \( \Psi \in \text{Aut}(D_b^G(M)) \) and \( \Phi \in \text{Aut}(D_b^H(N)) \). Then

\[
T_{S \circ \Psi} \cong T_S , \quad P_{F \circ \Psi} \cong P_F , \quad T_{\Phi \circ S} \cong \Phi \circ T_S \circ \Phi^{-1} , \quad P_{\Phi \circ F} = \Phi \circ P_F \circ \Phi^{-1} ;
\]

see [AA13, Proposition 13] and [6, Lemma 2.3].

#### 7.5.2 The case \( n = 1 \) and comparison to [CL12]

In section 7.2 we defined the functors \( H_{l,n}: D_b^G(X \times X^\ell) \to D_b(X^{n+\ell}) \) for \( n \geq 2 \). Regarding (7.11) it is a natural extension to the case \( n = 1 \) to set

\[
H_{l,1}^0 := \text{Inf}_{\Delta^1}^\mathbb{E} \circ D_b^G(X \times X^\ell) \to D_b^\mathbb{E}_{\ell+1}(X^{\ell+1}) .
\]

While the functors \( H_{l,n} \) for \( n \geq 2 \) are \( \mathbb{P}^{n-1} \)-functors (in the surface case), the functor \( H_{l,1}^0 \) is a \( \mathbb{P}^\ell \)-functor (for \( \dim X \) arbitrary) as follows from the following general observation.
Let $G$ be a finite group and $H \leq G$ a subgroup such that there is an element $g \in G$ of order $n = [G : H]$ such that $1, g, \ldots, g^{n-1}$ forms a system of representatives of the right cosets. Let $G$ act on a variety $M$. Recall that, in this case, the inflation functor is given by

$$
\operatorname{Inf} := \operatorname{Inf}^G_B : D^b_H(M) \to D^b_G(M), \quad \operatorname{Inf}(A) = \bigoplus_{k=0}^{n-1} g^k A \quad (7.52)
$$

with the linearisation of $\operatorname{Inf}(A)$ given by permutation of the summands.

**Lemma 7.5.1.** The inflation functor $\operatorname{Inf}$ is a $\mathbb{P}^{n-1}$-functor with $\mathbb{P}$-cotwist $g^*$.

*Proof.* The left and right adjoint of $\operatorname{Inf}$ is the restriction functor $\operatorname{Res}$. By (7.52) we indeed have $\operatorname{Res} \circ \operatorname{Inf} = \operatorname{id} \oplus g^* \cdots \oplus g^{(n-1)*}$. Condition (iii) of a $\mathbb{P}^{n-1}$-functor amounts to the fact that $\operatorname{Res} \cong g^{(n-1)*} \operatorname{Res}$. The counit map $\varepsilon : \operatorname{Inf} \circ \operatorname{Res}(B) = \bigoplus_{k=0}^{n-1} g^k B \to B$ for $(B, \lambda) \in D^b_G(M)$ is given by the components $\lambda^{-1} : g^k B \to B$; compare [Ela14, Section 3]. Using this, one can compute that the monad structure has the desired form.

However, the induced twists are not very interesting.

**Lemma 7.5.2.** For $n = [G : H] = 2$ the spherical twist $T_F$ associated to $F = \operatorname{Inf}$ equals the autoequivalence $M_a[1] := (\_ \otimes a)[1]$. For arbitrary $n \geq 2$ the $\mathbb{P}$-twist $P_F$ equals $[2]$.

*Proof.* Let $n = 2$ and $(B, \lambda) \in D^b_G(M)$. Let $\varphi : B \otimes a \to \operatorname{Inf} \operatorname{Res}(B)$ be the $G$-equivariant morphism with components $\operatorname{id} : B \to B$ and $-\lambda g : B \to g^* B$. This gives the exact triangle

$$
B \otimes a \xrightarrow{\varphi} \operatorname{Inf} \operatorname{Res}(B) \xrightarrow{\varepsilon} B.
$$

Since $T_F$ is defined by fitting into the exact triangle $\operatorname{Inf} \operatorname{Res}(B) \xrightarrow{\varepsilon} B \to T_F$, it follows that $T_F \cong M_a[1]$. Still for $n = 2$ the assertion for the $\mathbb{P}$-twist follows by the fact that $P_F = T_F^2$. For $n \geq 3$ one has to do calculations involving the double cone construction (7.48) of $P_F$ in order to show the assertion. We omit them.

Analogously to Section 7.1.3, for $E \in D^b(X)$ we consider

$$
H^{0}_{\ell,1}(E) := H^{0}_{\ell,1} \circ I_E : D^b_{S_{\ell}}(X^\ell) \to D^b_{S_{\ell+1}}(X^{\ell+1}), \quad A \mapsto \operatorname{Inf}^{\ell+1}_{\ell} (E \boxtimes A).
$$

Let now $X$ be a minimal resolution of the Kleinian singularity $\mathbb{C}^2/\Gamma$ for $\Gamma \subset \text{SL}(2, \mathbb{C})$ a finite subgroup so that we are in the situation of the classical McKay correspondence. Hence there is an equivalence $D^b(X) \cong D^b_{\ell}((\mathbb{C}^2)$. For $V_i$ an irreducible representation of $\Gamma$, consider $E_i = C(0) \otimes V_i \in D^b_{\ell}((\mathbb{C}^2)$ which corresponds to the line bundle $O(-1)$ on a component of the exceptional divisor of $X$. Then the $H^{0}_{\ell,1}(E_i)$ are exactly the functors $P_1(\ell)$ of [CL12] which give rise to a categorical action of the Heisenberg algebra on the derived categories of the Hilbert schemes of points on $X$. For higher $n$ the construction in [CL12] does not give explicit lifts of the Nakajima operators $q_n$. Instead, functors $P^{(n)}_i(\ell)$ are constructed which correspond to other generators of the Heisenberg algebra than the Nakajima operators do; see [CL12, Section 8.2]. The functors $P^{(n)}_i(\ell)$ are given in terms of our notation by

$$
P^{(n)}_i(\ell) : D^b_{\ell}(X^\ell) \to D^b_{\ell+n}(X^{n+\ell}), \quad A \mapsto \operatorname{Inf}^{\ell+n+\ell}_{\ell+n} (E^{\ell+n} \boxtimes A)
$$

which is a direct summand of $H^{0}_{n+\ell-1,1}(E_i) \circ \cdots \circ H^{0}_{1,1}(E_i) \circ H^{0}_{0,1}(E_i)$. In contrast, $H^{0}_{\ell,1}(E)$ is given by $A \mapsto \operatorname{Inf}^{\ell+n+\ell}_{\ell,n} (E \boxtimes A)$ where $E_\Delta$ denotes the push-forward of $E$ along the small diagonal of $X^\ell$. 

202
7.5.3 Induced autoequivalences on the Hilbert schemes

Let $X$ be a smooth projective surface and $m \geq 2$. We will mostly omit the Bridgeland–King–Reid–Haiman equivalence $\Phi_m : \mathcal{D}^b(X^{[m]}) \xrightarrow{\cong} \mathcal{D}^b_{\mathcal{E}_m}(X^m)$ in the notation and interpret every functor between the equivariant derived categories of the cartesian product as one between the derived categories of the Hilbert schemes and vice versa. For $m$ even we set $r = \frac{m}{2} - 1$ and for $m$ odd we set $r = \frac{m-1}{2}$. By Theorem C there are the $\mathbb{P}^{m-\ell-1}$-functors $H_{\ell,m-\ell} : \mathcal{D}^b(X \times X^{[\ell]}) \to \mathcal{D}^b(X^{[m]})$ for $\ell = 0, \ldots, r$. We denote the associated $\mathbb{P}$-twists by $P_{\ell,m-\ell} = : P_{H_{\ell,m-\ell}} \in \text{Aut}(\mathcal{D}^b(X^{[m]}))$. Recall that the group of standard autoequivalences

$$\text{Aut}(\mathcal{D}^b(X^{[m]})) \supset \text{Aut}^s(\mathcal{D}^b(X^{[m]})) \cong \mathbb{Z} \times \left( \text{Aut}(X^{[m]}) \ltimes \text{Pic}(X^{[m]}) \right)$$

is the subgroup spanned by shifts, push-forwards along automorphisms and tensor products by line bundles. We consider the group

$$\text{Aut}^{s+H}(\mathcal{D}^b(X^{[m]})) := \langle \text{Aut}^s(\mathcal{D}^b(X^{[m]})), P_{0,m} \cdot \ldots \cdot P_{r,m-r} \rangle \subset \text{Aut}(\mathcal{D}^b(X^{[m]})).$$

The functors $H_{0,2}$ and $H_{1,2}$ are $\mathbb{P}^1$-functors, hence spherical. Accordingly, in the cases $m = 2, 3$ we replace $P_{0,2}$ and $P_{1,2}$ by their square roots $T_{0,2} := T_{H_{0,2}}$ and $T_{1,2} := T_{H_{1,2}}$ in the definition of $\text{Aut}^{s+H}(\mathcal{D}^b(X^{[m]}))$.

Let $\pi : X^m \to S^m X := X^m/\mathcal{G}_m$ be the quotient map. We write points of the symmetric product as formal sums of points of $X$. For $\nu = (\nu_1, \ldots, \nu_s)$ a partition of $m$ there is the stratum

$$X^m_{\nu} := \{ x \in X^m | \pi(x) = \nu_1 \cdot y_1 + \cdots + \nu_s \cdot y_s \text{ with pairwise distinct } y_i \in X \} \subset X^m.$$ 

We denote the complement of its closure by $\bar{X}^m_{\nu}$. For $1 \leq k \leq m$ set $\nu(k) := (k, 1, \ldots, 1)$ as a partition of $m$. Note that, for $r < k \leq m$ we have

$$X^m_{\nu(k)} = \left( \bigcup_{I \subset [m], |I| = k} \Delta_I \right) \setminus \left( \bigcup_{I \subset [m], |I| = k+1} \Delta_I \right), \quad \bar{X}^m_{\nu(k)} = X^m \setminus \left( \bigcup_{I \subset [m], |I| = k} \Delta_I \right). \quad (7.53)$$

For $x \in X^m$ we denote by $\text{orb}(x) \subset X^m$ the orbit of $x$ under the $\mathcal{G}_m$-action on $X^m$. We set

$$\bar{C}(x) := \text{orb}(x) \cap a_m \in \mathcal{D}^b_{\mathcal{E}_m}(X^m).$$

Lemma 7.5.3. For $r \leq \ell \leq m$ we have

$$P_{\ell,m-\ell}(\bar{C}(x)) \cong \begin{cases} \bar{C}(x)[-2(m-\ell-1)] & \text{ for } x \in \bar{X}^m_{\nu(m-\ell)}, \\ \bar{C}(x) & \text{ for } x \in X^m_{\nu(m-\ell)}. \end{cases}$$

Also,

$$T_{0,2}(\bar{C}(x)) \cong \begin{cases} \bar{C}(x)[-1] & \text{ for } x \in \Delta, \\ \bar{C}(x) & \text{ for } x \in X^2 \setminus \Delta \end{cases}; \quad T_{1,2}(\bar{C}(x)) \cong \begin{cases} \bar{C}(x)[-1] & \text{ for } x \in X^3_{\nu(2)}, \\ \bar{C}(x) & \text{ for } x \in \bar{X}^3_{\nu(2)}. \end{cases}$$

Proof. Every $x \in X^m_{\nu(m-\ell)}$ has a point of the from $y = (y_1, \ldots, y_\ell, y, \ldots, y)$ in its $\mathcal{G}_m$-orbit. Then $H_{\ell,m-\ell}(\bar{C}(y,y_1, \ldots, y_\ell)) \cong \bar{C}(y) \cong \bar{C}(x)$. By (7.53) it also follows that $H_{\ell,m-\ell}(\bar{C}(x)) = 0$ for $x \in \bar{X}^m_{\nu(m-\ell)}$. The assertion for the $\mathbb{P}$-twist follows by (7.49) and the assertion for the spherical twists follows by (7.50). \qed
Note that \( X^m_{\nu} \cup \bar{X}^m_{\nu} \subseteq X^m \) so that the above does not describe the value of the \( P_{\ell,m-\ell} \) on all skyscraper sheaves \( \mathcal{C}(x) \) of orbits. In fact, for \( x \in X^m \setminus (X^m_{\nu} \cup \bar{X}^m_{\nu}) \) the object \( P_{\ell,m-\ell}(\mathcal{C}(x)) \) is again supported on \( \text{orb}(x) \) but not simply a shift of \( \mathcal{C}(x) \).

**Proposition 7.5.4.** The abelianisation of the group \( \text{Aut}^{st+H}(\mathcal{D}^b(X^m)) \) is given by

\[
\text{Aut}^{st+H}(\mathcal{D}^b(X^m))_{ab} \cong \mathbb{Z} \times (\text{Aut}(X^m) \times \text{Pic}(X^m))_{ab} \times \mathbb{Z}^{r+1}.
\]

**Proof.** Let \( \Psi = M_L \circ \varphi \circ P_{0,m}^{a_0} \circ \cdots \circ P_{r,m-r}^{a_r}[b] \) for \( L \in \text{Pic}(X^m) \), \( \varphi \in \text{Aut}(X^m) \), and \( a_0, \ldots, a_r, b \in \mathbb{Z} \). We have to show that \( \Psi = \text{id} \) implies \( L \cong \mathcal{O}_X \), \( \varphi = \text{id} \), and \( a_0 = \cdots = a_r = b = 0 \). Let \( \xi = \{x_1, \ldots, x_m\} \subset X \) be a reduced subscheme of length \( m \). Under the BKRH equivalence \( \mathcal{C}([\xi]) \) corresponds to \( \mathcal{C}(x) \) with \( x = (x_1, \ldots, x_m) \). Since \( x \in X^m_{\nu(\ell)} \) for all \( k \geq 2 \) we get by the previous lemma \( \Psi(\mathcal{C}([\xi])) = \mathcal{C}([\varphi(\xi)][b]) \) which implies \( b = 0 \) and \( \varphi = \text{id} \). Let now \( x \in X^m_{\nu(\ell)} \) and \( A = \Phi^{-1}_{m}(\mathcal{C}(x)) \). Again by the previous lemma, \( \Psi(A) = L \otimes A^{-1}(-2(m-r-1)a_r) \) which shows \( a_r = 0 \). Testing inductively the values of \( \Psi(\mathcal{C}(x)) \) for \( x \in X^m_{\nu(\ell)} \) shows that also \( a_{r-1} = \cdots = a_0 = 0 \) hence \( \Psi = M_L \). Finally \( \Psi(\mathcal{O}_{X^m}) = L \) shows \( L \cong \mathcal{O}_{X^m} \). The proof goes through in the same way in the cases \( m = 2, 3 \) where \( P_{0,2} \) and \( P_{1,2} \) have to be replaced by \( T_{0,2} \) and \( T_{1,2} \). \( \square \)

**Remark 7.5.5.** Even before taking the abelianisation, the \( P_{\ell,m-\ell} \) commute with a large class of autoequivalences namely with those induced by automorphisms and line bundles on the surface \( X \). For the proof in the case \( \ell = 0 \) see [6, Lemma 5.4]. The proof for arbitrary \( \ell \) is similar.

**Lemma 7.5.6.** Every autoequivalence in \( \text{Aut}^{st+H}(\mathcal{D}^b(X^m)) \) is rank-preserving up to sign and sends objects that are supported on a non-trivial subset of \( X^m \) to objects with the same property.

**Proof.** For \( \ell = 0, \ldots, r \) and every object \( A \in \mathcal{D}^b_{\Omega^\ell}(X \times X^\ell) \) the object \( H_{\ell,m-\ell}(A) \) is supported on the closure of \( X^m_{\nu(\ell)} \) hence has rank zero. By the double cone construction (7.48) of the \( \mathbb{P} \)-twist it follows that \( \text{rank} P_{\ell,m-\ell}(B) = \text{rank} B \) for all \( B \in \mathcal{D}^b(X^m) \). Also every standard autoequivalence is rank preserving with the exception of odd shifts which multiply the rank by \(-1 \). The argument for the second assertion is similar. \( \square \)

There are plenty of examples, especially in the case that \( X \) is a K3 or Enriques surface, of autoequivalences in \( \mathcal{D}^b(X^m) \) which are not rank preserving or send skyscraper sheaves to objects whose support is the whole \( X^m \). By the previous lemma we know that these autoequivalences are not contained in \( \text{Aut}^{st+H}(\mathcal{D}^b(X^m)) \). Here is a list of those examples the author is aware of:

1. For every non rank-preserving autoequivalence \( \Psi \in \text{Aut}(\mathcal{D}^b(X)) \) on the surface, Ploog’s construction [Plo07, Section 3.1] gives an autoequivalence \( \Psi[n] \in \text{Aut}(\mathcal{D}^b(X^m)) \) which is non rank-preserving too.

2. For \( X \) a K3 surface, the universal ideal functor \( \mathcal{F} = FM_{I_S} : \mathcal{D}^b(X) \rightarrow \mathcal{D}^b(X^m) \) is a \( \mathbb{P}^{m-1} \)-functor and the induced twist \( P_{\mathcal{F}} \) sends skyscraper sheaves to objects whose support is \( X^m \); see [Add16] and [6, Lemma 5.7].
3. There is a variant \( G : \mathcal{D}^b(X) \to \mathcal{D}^b(X^{[m]}) \) of \( F \) called the truncated universal ideal functor; see [5] or Section 7.5.5. For \( X \) a K3 surface, it is again a \( \mathbb{P}^{m-1} \)-functor that sends skyscraper sheaves to objects whose support is \( X^{[m]} \).

4. For \( X \) an Enriques surface, the universal ideal functor \( \mathcal{F} \mathcal{M}_{X} : \mathcal{D}^b(X) \to \mathcal{D}^b(X^{[m]}) \) is neither spherical nor a \( \mathbb{P} \)-functor but fully faithful. Nevertheless, there is an induced autoequivalence which is not rank-preserving; see [5].

5. For \( X \) an Enriques surface also the truncated universal ideal functor is fully faithful and induces another non-rank-preserving autoequivalence.

6. For \( X \) a K3 surface, the structural sheaf \( \mathcal{O}_{X^{[m]}} \) is a \( \mathbb{P}^m \)-object and the induced twist \( P_{\mathcal{O}} \) sends skyscraper sheaves to objects whose support is \( X^{[m]} \). The same holds for twists along general \( \mathbb{P}^n \)-objects which are supported on the whole \( X^{[m]} \). Spherical objects in \( \mathcal{D}^b(X) \) induce \( \mathbb{P}^m \)-objects in \( \mathcal{D}^b(X^{[m]}) \); see [PS14].

7. For \( X \) an Enriques surface, the structural sheaf \( \mathcal{O}^{[m]}_{X} \) is an exceptional object and induces an autoequivalence which is not rank-preserving. The same holds for any exceptional object of non-zero rank. Exceptional objects in \( \mathcal{D}^b(X) \) induce exceptional objects in \( \mathcal{D}^b(X^{[m]}) \); see [5].

8. For \( X = A \) an abelian surface the pull-back \( \Sigma^* : \mathcal{D}^b(A) \to \mathcal{D}^b_{\mathbb{P}^m}(A^m) \cong \mathcal{D}^b(A^{[m]}) \) along the summation morphism is a \( \mathbb{P}^{m-1} \)-functor with \( \mathbb{P} \)-cotwist \([-2] \); see [Mea15]. Let \( \Delta \subset A^m \) be the small diagonal. Then \( \text{supp}(P_{\Sigma^*}(\mathcal{O}_\Delta)) = A^m \). Indeed, the composition \( \Sigma \circ \delta : A \to A \) with the embedding of the small diagonal is multiplication by \( m \). Hence, \( \Sigma^*\mathcal{O}_\Delta \) is a locally free sheaf of rank \( m \). Thus, also \( \Sigma^*\Sigma^*\mathcal{O}_\Delta \) is a locally free sheaf of rank \( m \) and the claim follows by the double cone construction (7.48) of the \( \mathbb{P} \)-twist.

### 7.5.4 Induced autoequivalences on the Kummer varieties

Recall that for \( A \) an abelian surface there is the equivalence \( \hat{P}_{m} : \mathcal{D}^b(K_{m-1}A) \cong \mathcal{D}^b_{\mathbb{P}^m}(N_{m-1}A) \); see e.g. [Mea15, Lemma 6.2]. Thus, the \( \mathbb{P} \)-functors of Theorem C' can be considered as \( \mathbb{P} \)-functors to the derived category of the generalised Kummer variety. For the induced twists \( \hat{P}_{t,m-r} := P_{\hat{H}_{t,m-r}} \) the picture is very similar to the Hilbert scheme case with the following exception. The \( \mathbb{P}^{m-1} \)-functor \( \hat{H}_{0,m} : \mathcal{D}^b(A_m) \to \mathcal{D}^b_{\mathbb{P}^m}(N_{m-1}A) \) splits into the \( \mathbb{P}^{m-1} \)-objects \( \mathbb{C}(a, \ldots, a) = \mathbb{C}(a, \ldots, a) \otimes a_m \in \mathcal{D}^b_{\mathbb{P}^m}(N_{m-1}A) \) for \( a \in A_m \); see [6, Section 6]. Thus, it gives rise to \( m^4 \) different autoequivalences \( \hat{P}_a := P_{\mathbb{C}(a, \ldots, a)} \) such that \( \prod_{a \in A_m} \hat{P}_a \cong \hat{P}_{0,m} \). We define \( \text{Aut}^{st+H}(\mathcal{D}^b(K_{m-1}A)) \subset \text{Aut}(\mathcal{D}^b(K_{m-1}A)) \) by

\[
\text{Aut}^{st+H}(\mathcal{D}^b(K_{m-1}A)) := \langle \text{Aut}^{st}(\mathcal{D}^b(K_{m-1}A)), \hat{P}_a : a \in A_m, \hat{P}_{1,m-1}, \ldots, \hat{P}_{r,m-r} \rangle.
\]

Analogous to Proposition 7.5.4 we get

\[
\text{Aut}^{st+H}(\mathcal{D}^b(K_{m-1}A))_{ab} \cong \mathbb{Z} \times (\text{Aut}(K_{m-1}A) \ltimes \text{Pic}(K_{m-1}A))_{ab} \times \mathbb{Z}^{m^4} \times \mathbb{Z}^r. \tag{7.54}
\]

Again, all the autoequivalences in the group \( \text{Aut}^{st+H}(\mathcal{D}^b(K_{m-1}A)) \) are rank-preserving up to sign and preserve objects with non-trivial support. Thus, neither the \( \mathbb{P} \)-twist induced by the universal ideal functor (see [Men15]) nor twists induced by \( \mathbb{P}^{m-1} \)-objects whose support is the whole \( K_{m-1}A \), for example \( \mathcal{O}_{K_{m-1}A} \), are contained in \( \text{Aut}^{st+H}(\mathcal{D}^b(K_{m-1}A)) \).
7.5.5 Relation to the (truncated) universal ideal functors

Let $X$ be a smooth quasi-projective surface and $F = F_m = FM_{\mathbb{T}_2} : D^b(X) \to D^b(X^m)$ the universal ideal functor. It follows from results of [Sca09] that $\Phi \circ F \cong FM_K$ where $K \in D^b_{\mathbb{E}_m}(X \times X^m)$ is the complex concentrated in degrees $[0, m]$ given by

$$0 \to \mathcal{O}_{X \times X^m} \to \bigoplus_{i=0}^{m} \mathcal{O}_{D_i} \to \bigoplus_{|I|=2} \mathcal{O}_{D_I} \otimes \mathfrak{a}_I \to \bigoplus_{|I|=3} \mathcal{O}_{D_I} \otimes \mathfrak{a}_I \cdots \to \mathcal{O}_{D_{[m]}} \otimes \mathfrak{a}_{[m]} \to 0;$$

see [Mea15, Section 6]. We define the truncated universal ideal functor as

$$G := G_m := FM_{\sigma \leq 1K} : D^b(X) \to D^b_{\mathbb{E}_m}(X^m),$$

$$\sigma \leq 1K = (0 \to \mathcal{O}_{X \times X^m} \to \bigoplus_{i=0}^{m} \mathcal{O}_{D_i} \to 0).$$

We have $G^R G \cong F^R F$; see [5, Section 5]. It follows that $G$ is again a $\mathbb{P}^{m-1}$-functor if $X$ is a K3 surface and fully faithful if $X$ is an Enriques surface or more generally a surface with $p_g = 0 = q$. In the following we perform the computations at the level of the functors since we find it a bit easier and more intuitive. However, in order to show that the induced maps between the functors take the expected form one has to do the calculation at the level of the FM kernels what we omit. We have

$$FM_{K^0} \cong H^*(\_ \otimes \mathcal{O}_{X^m}), \quad FM_{K^1} \cong \text{Inf}_{\mathbb{E}_{m-1}} \circ \text{pr}_1 \circ \text{triv},$$

$$FM_{K^0} \cong \text{Inf}_{\mathbb{E}_m \times \mathbb{E}_{m-q}} \circ \delta_{[q]}^* \circ \text{pr}_1^* \circ \text{triv} \quad \text{for } q = 2, \ldots, m \quad (7.56)$$

where for $I \subset [m]$ we denote by $p_I : \Delta_I \cong X \times X^I \to X$ the projection to the first factor. For the following, consider $\text{pr}_X^* \circ \text{triv} : D^b(X) \to D^b_{\mathbb{E}_{m-2}}(X \times X^{m-2})$.

**Lemma 7.5.7.** $H^m_{m-2, 2} \circ \text{pr}_X^* \circ \text{triv} \cong FM_{\sigma \geq 2K[2]}$.

**Proof.** Under the isomorphism $\Delta_{[2]} \cong X \times X^{m-2}$ we have $\text{pr}_X = p_2$. Moreover, for $i \geq 1$ under the identification $\Delta_{[i+2]} \cong X \times X^{m-2}$ we have $\text{pr}_X \circ \iota_{[i]} = p_{i+2}$. Now it follows from comparing $(7.10)$ and $(7.56)$ that $H^i_{m-2, 2} \circ \text{pr}_X^* \circ \text{triv} \cong FM_{K^{i+2}}$ for every $i = 0, \ldots, m - 2$. \qed

**Lemma 7.5.8.** $H^L_{m-2, 2} \circ G \cong \text{pr}_X^* \circ \text{triv}[-1]$.

**Proof.** We set $\ell = m - 2$. Note that the left adjoints $H^L_{\ell, 2}$ are given by $(7.13)$ with $\iota_{[i]}^*$ replaced by $\iota_{[i]}!$ and $\delta_{[2+i]}^*$ replaced by $\delta_{[2+i]}$! We have

$$(\_)^{E_{2+i}} M_{a_{2+i}} \delta_{[2+i]}^* \text{Res}(\mathcal{O}_{X^m}) \cong (\mathcal{O}_{X \times X^\ell-i} \otimes a_{2+i})^{E_{2+i}} = 0$$

which shows that $H^L_{2,m-2} \circ FM_{K^0} = 0$ for all $0 \leq i \leq m - 2$. Let $q_a : X \times X^{\ell-i} \to X$ denote the projection to the $a$-th factor of $X^{\ell-i}$. For $E \in D^b(X)$ we have by Section 7.3.5

$$(\_)^{E_{2+i}} M_{a_{2+i}} \delta_{[2+i]}^* \text{Res}(\bigoplus_{i=1}^{m} \text{pr}_i^* E)$$

$$\cong \bigoplus_{a=1}^{\ell-i} (a_{2+i} \otimes q_a^* E)^{E_{2+i}} \oplus (E \otimes a_{2+i-1} \otimes \mathcal{O}_{X^{\ell-i}})^{E_{2+i-1}} \cong \begin{cases} 0 & \text{for } i > 0; \\ \text{pr}_X^* E & \text{for } i = 0. \end{cases}$$

Hence, $H^L_{m-2, 2} \circ G \cong H^L_{m-2, 2} \circ FM_{K^1}[-1] \cong H^L_{m-2, 2} \circ FM_{K^1}[-1] = \text{pr}_X \circ \text{triv}[-1]$. \qed
For $m \geq 4$ the functors $H_{m-2,2}$ are not spherical; compare Section 7.3.9. Nevertheless, one can consider the associated twist $T_{m-2,2} := \text{cone}(H_{m-2,2}H^R_{m-2,2} \to \text{id})$ as well as its left adjoint $T^L_{m-2,2} = \text{cone}(\text{id} \to H_{m-2,2}H^L_{m-2,2})$ though they are not autoequivalences for $m \geq 4$.

**Proposition 7.5.9.** $T^L_{m-2,2} \circ G \cong F$ for $m \geq 2$.

**Proof.** The previous two lemmas combined give $H_{m-2,2} \circ H^L_{m-2,2} \circ G \cong \text{FM}_{\sigma \geq 2}K_{[1]}$. It follows that $T^L_{m-2,2} \circ G \cong \text{cone}(\text{FM}_{\sigma \geq 1}K \to \text{FM}_{\sigma \geq 2}K_{[1]}) \cong \text{FM}_K \cong F$. \hfill \Box

In the cases $m = 2, 3$ the twists $T_{m-2,2}$ and $T^L_{m-2,2} = T^{-1}_{m-2,2}$ are autoequivalences. Thus, the above proposition gives another proof that $G^R G \cong F^R F$ in these cases. Furthermore, by (7.51) we get

**Corollary 7.5.10.** For $X$ a K3 surface, we have the relations $T_{0,2}T_{0,2} = T_{G_2}T_{0,2}$ and $T_{1,2}P_{F_3} = P_{G_3}T_{1,2}$ in $\text{Aut}(D^b(X^2))$ and $\text{Aut}(D^b(X^3))$, respectively.

We get similar relations for $X$ an Enriques surface by [5, Remark 3.12]. Let now $A$ be an abelian surface and $\hat{F}_m = \hat{F} = \text{FM}_{I_{\hat{K}}} : D^b(A) \to D^b(K_{m-1}A)$ the universal ideal functor. It is a $\mathbb{P}^{m-2}$-functor for $m \geq 3$ and a $\mathbb{P}^1$-functor for $m = 2$; see [Mea15] and [3]. Then $\hat{F} \circ \hat{F} = \text{FM}_{\hat{K}}$ where $\hat{K}$ is the complex

$$\hat{O}_{\hat{K}} = \bigoplus_{i=0}^{m-2} \hat{O}_{D_i} \to \bigoplus_{|I|=2} \hat{O}_{D_I} \hat{a}_I \to \bigoplus_{|I|=3} \hat{O}_{D_I} \hat{a}_I \cdots \to \hat{O}_{D_{m}} \hat{a}_m \to 0;$$

where $\hat{D}_I = D_I \cap (A \times N_{m-1}A)$. Following the computations in [Mea15, Section 6] one sees that the truncated version $\hat{G}_m := \hat{G} := \text{FM}_{\hat{I}_{\hat{K}}}$ again satisfies $\hat{G} \circ \hat{G}^R \cong \hat{F}^R \circ \hat{F}$. Hence $\hat{G}$ is again a $\mathbb{P}^{m-2}$ functor for $m \geq 3$ and a $\mathbb{P}^1$-functor for $m = 2$. Now, we can prove analogously to above that $\hat{T}^L_{m-2,2} \circ \hat{F}_m = \hat{G}_m$. The key step is again to compute that

$$\hat{T}^L_{m-2,2} \circ \hat{G}_m \cong \hat{p}X \circ \text{triv}[-1], \quad \hat{p}X = (M_{m-2,2}A \leftrightarrow A \times A^{m-2'} \dashrightarrow \text{pr}_X, A).$$

**7.5.6 Braids on hyperkähler fourfolds**

We say that two elements $a, b$ of a group satisfy the **braid relation** if $aba = bab$. Two twists $T_E, T_F$ along spherical objects satisfy the braid relation if $\text{Hom}^*(E, F) = \mathbb{C}[n]$ for some $n \in \mathbb{Z}$; see [ST01]. There is the following straightforward generalisation which gives a criterion for twists along spherical functors to satisfy the braid relation; compare [AL13, Theorem 1.2].

**Proposition 7.5.11.** Let $G = \text{FM}_G, H = \text{FM}_H : D^b(M) \to D^b(N)$ be two spherical functors such that $G^R H \cong \text{id}$ and $\text{Hom}_{D^b(M \times N)}(G, H) = \mathbb{C}$. Let $F = \text{cone}(G \xrightarrow{\psi} H)$ for $0 \neq \psi \in \text{Hom}(G, H)$ and set $F = \text{FM}_F$. Then $F$ is also a spherical functor and every pair of $T_F, T_G, T_H$ spans $\langle T_F, T_G, T_H \rangle$ and satisfies the braid relation.

**Proof.** Composing the triangle $GG^R \to \text{id} \to T_G$ with $H$ and using that $G^R H \cong \text{id}$ we get the triangle $G \to H \to T_G H$. The map $G \to H$ of this triangle is non-zero. Indeed, otherwise we had $T_G H = H \oplus G[1]$ contradicting the fact that all three functors are spherical. Because of $\text{Hom}(G, H) = \mathbb{C}$ it follows that $F \cong T_G H$. This shows that $F$ is spherical and by (7.51) there is the relation

$$T_F = T_G \circ T_H \circ T_G^{-1}$$

(7.58)
in $\text{Aut}(D^b(N))$. Taking left adjoints of $G^R H \cong \text{id}$ we get $H^L G \cong \text{id}$. This gives the triangle $T_H^{-1} G \to G \to H$ which shows that $T_H^{-1} G = F[-1]$ and

$$T_F = T_H^{-1} \circ T_G \circ T_H.$$  \hspace{1cm} (7.59)

The assertion follows from the equations (7.58) and (7.59).

If $M$ and $N$ are projective we have $\text{Hom}_{M \times N}(G, H) \cong \text{Hom}_{M \times M}(O_{\Delta}, G^R \ast H)$. This follows from the fact that the unit and counit of the adjunction are already defined on the level of the FM kernels; see [CW10] or [AL12]. Thus, the second assumption of the proposition, namely $\text{Hom}_{D^b(M \times N)}(G, H) = \mathbb{C}$, follows form the first assumption, namely $G^R H \cong \text{id}$.

**Proposition 7.5.12.** 1. Let $X$ be a $K3$ surface. Then every pair of $T_{F_2}, T_{G_2}, T_{O_{2}}$ spans the subgroup $\langle T_{F_2}, T_{G_2}, T_{O_{2}} \rangle \subset \text{Aut}(D^b(X^3))$ and satisfies the braid relation.

2. Let $A$ be an abelian surface. Then every pair of $T_{F_3}, T_{G_3}, T_{1,2}$ spans the subgroup $\langle T_{F_3}, T_{G_3}, T_{1,2} \rangle \subset \text{Aut}(D^b(K_2 A))$ and satisfies the braid relation.

**Proof.** We apply the above proposition to $G = G_2$ and $H = H_{0,2}[-1]$. By Lemma 7.5.8 we have $H^L G = \text{id}$ and hence also $G^R H \cong \text{id}$. Thus, the assumptions of Proposition 7.5.11 are fulfilled (one either has to apply the BKRH equivalence or make the straight-forward generalisation of Proposition 7.5.11 to equivariant spherical FM transforms).

In the second case we must consider $G = G_3$ and $H = H_{1,2}[-1]$. Note that we have $M_{1,2} A \cong A$ so that (7.57) gives $H^L G = \text{id}$. \hfill $\Box$

### 7.5.7 Semi-orthogonal decompositions in the curve cases

Let $C$ be a smooth curve. As stated in Corollary B, it follows from Proposition A that there is the semi-orthogonal decomposition

$$D^b_{\mathcal{S}_m}(C^m) = \langle A_{0,m}, A_{1,m-1}, \ldots, A_{r,m-r}, B \rangle, A_{\ell,m-\ell} = H_{\ell,m-\ell}(D^b_{\mathcal{S}_m}(C \times C^\ell)).$$ (7.60)

Since the symmetric product $S^m C$ is smooth, the pull-back along $\pi: C^m \to S^m C$ maps to the bounded derived category, i.e. $\pi_* \text{triv}: D^b(S^m C) \to D^b_{\mathcal{S}_m}(C^m)$. Note that this holds exclusively for curves. Since $(\pi_* O_{S^m C}) = O_{S^m C}$, it follows by the projection formula that $(\_)^{S^m} \pi_* \pi^* \text{triv} \cong \text{id}$ which means that $\pi^* \text{triv}$ is fully faithful. For $I \subset \{m\}$ with $|I| \geq 2$ we have $(\_)^{S^I} M_{a_I} \delta_I \text{Res} \pi^* \text{triv} = 0$. Hence, $H^L_{\ell,m-\ell} \pi^* \text{triv} = 0$ for all $\ell \geq 2$ which shows that $\pi^* \text{triv}(D^b(S^m C)) \subset B$.

Also note that the semi-orthogonal decomposition (7.60) can be further refined by considering for $\ell \leq r$ the semi-orthogonal decompositions of $D^b_{\mathcal{S}_r}(C \times C^\ell)$ induced by those of $D^b_{\mathcal{S}_r}(C^\ell)$ which are themselves of the form (7.60).

### 7.5.8 Induced autoequivalences in the curve cases

**Lemma 7.5.13.** Let $X$ be a smooth variety of odd dimension. Then

$$S_X^{-(n-1)} H_{\ell,n}^L M_{\delta_{n+\ell}} \cong M_{\delta_{n}} H_{\ell,n}^R$$ for $n \geq 2$.\hfill 208
Proof. There is the following crucial difference between the odd and the even dimensional case. If $X$ is even dimensional, the $\mathbb{S}_m$-equivariant canonical bundle of $X^m$ equals $\omega_{X^m} \cong \omega_X^{\otimes m}$ where the linearisation is the one acting by permuting the factors, while in the odd dimensional case the linearisation is given by $\omega_{X^m} \cong \omega_X^{\otimes m} \otimes a_m$. Indeed, on the fibre $\omega_X(x)$ the stabiliser of $x \in X^m$ acts by permuting blocks of length $d = \text{dim} X$ in the wedge product.

With this in mind the proof is the same as the proof that $\bar{S}_X^{(n-1)} H^{R,\tau}_{\ell,n} \cong H^{R,\tau}_{\ell,n}$ for $X$ even dimensional; see Section 7.2.5. □

Let now $X = C$ be a smooth curve. Then $H_{\ell,n} : D^b_{\mathbb{G}_m}(C \times C^\ell) \to D^b_{\mathbb{G}_m}(X^{n+\ell})$ is fully faithful for $n > \min\{\ell, 1\}$; see Proposition A. Let $m = n + \ell$ and $\mathfrak{A}_m \subset \mathbb{S}_m$ be the alternating group. The functor $\text{Res} : D^b_{\mathbb{G}_m}(X^m) \to D^b_{\mathfrak{A}_m}(X^m)$ is spherical with cotwist $M_a$ and twist $\tau^*[1]$ where $\tau$ is any element of $\mathfrak{S}_m \setminus \mathfrak{A}_m$. This follows by the Lemmas 7.5.1 and 7.5.2 together with the fact that a functor $F$ is spherical if and only if $F^R$ is spherical with the roles of the twist and cotwists exchanged; see [AL13, Theorem 1.1]. The composition $\text{Res}_{\mathfrak{S}_m}^{\mathfrak{A}_m+\ell} \circ H_{\ell,n}$ is again a spherical functor. This follows by [HS16, Theorem 4.13], whose assumptions are fulfilled by Lemma 7.5.13. We have $\tau^* \circ \text{Res}_{\mathfrak{S}_m}^{\mathfrak{A}_m+\ell} \circ H_{\ell,n} \cong \text{Res}_{\mathfrak{S}_m}^{\mathfrak{A}_m+\ell} \circ H_{\ell,n}$ for $\tau \in \mathfrak{S}_m \setminus \mathfrak{A}_m$. Thus, the spherical twist $\tilde{T}_{\ell,n} := T_{\text{Res} \circ H_{\ell,n}} \in \text{Aut}(D^b_{\mathfrak{S}_m}(X^m))$ is equivariant, that means $\tau^* \circ T_{\ell,n} \cong T_{\ell,n} \circ \tau^*$, by (7.51). Hence, there is an induced autoequivalence $T_{\ell,n} \in \text{Aut}(D^b_{\mathfrak{S}_m}(X^m))$; see [Pl07, Theorem 6]. By the same arguments, the fully faithful functors $\tilde{H}_{\ell,n} : D^b_{\mathfrak{S}_m}(M_{\ell,n} E) \to D^b_{\mathfrak{S}_m}(N_{\ell,n+\ell-1} E)$ for $E$ an elliptic curve induce autoequivalences of $D^b_{\mathfrak{A}_m}(N_{\ell,n+\ell-1} E)$ and $D^b_{\mathfrak{S}_m+\ell}(N_{\ell,n+\ell-1} E)$.

One can also construct $T_{\ell,n} \in \text{Aut}(D^b_{\mathfrak{S}_m}(C^m))$ directly as the cone

$$T_{\ell,n} = \text{cone}(H_{\ell,n} H^R_{\ell,n} \oplus M_a H_{\ell,n} H^R_{\ell,n} M_a \to \text{id})$$

where $c$ is the direct sum of the counit maps. It follows by Lemma 7.5.13 that

$$T_{\ell,n} H_{\ell,n} \cong M_{\mathfrak{A}_m+\ell} H_{\ell,n} M_a \tilde{\sigma}_{C}^{-(n-1)}[1], \quad T_{\ell,n} M_{\mathfrak{A}_m+\ell} H_{\ell,n} \cong H_{\ell,n} M_a \tilde{\sigma}_{C}^{-(n-1)}[1],$$

and $T_{\ell,n}(B) = B$ for $B \in \ker H^R_{\ell,n} \cap \ker(H^R_{\ell,n} M_a)$. Thus, similarly to Lemma 7.5.3

$$T_{\ell,n}(\tilde{C}(x)) = \begin{cases} \tilde{C}(x)[-n-2] & \text{for } x \in X^m_{\nu(n)} \\ \tilde{C}(x) & \text{for } x \in X^m_{\nu(n)}. \end{cases} \quad (7.61)$$

7.5.9 Some conjectures

As we see from the list at the end of Section 7.5.3, there are surfaces $X$ for which the subgroup

$$\text{Aut}^{st+H}(D^b(X^m)) = \langle \text{Aut}^{st}(D^b(X^m)), P_{0,m}, \ldots, P_{r,m-r} \rangle \subset \text{Aut}(D^b(X^m))$$

generated by standard autoequivalences and twists along the Nakajima P-functors is far smaller than the full group $\text{Aut}(D^b(X^m))$. However, in the case that the canonical bundle of the surface is ample or anti-ample there are no non-standard autoequivalences coming from the surface (see [BO01]) and we expect the following to hold.

Conjecture 7.5.14. Let $X$ be a smooth projective surface with $\omega_X$ ample or anti-ample. Then for $m = 2, 3$ we have $\text{Aut}^{st+H}(D^b(X^m)) = \text{Aut}(D^b(X^m))$, i.e. $\text{Aut}(D^b(X^2)) = \langle \text{Aut}^{st}(D^b(X^2)), T_{0,2} \rangle$, $\text{Aut}(D^b(X^3)) = \langle \text{Aut}^{st}(D^b(X^3)), P_{0,3}, T_{1,2} \rangle$. 

209
For \( m \geq 4 \), we expect there to be further autoequivalences \( r_{m-r-1}, \ldots, p_{m-3}, t_{m-2} \) with the same behaviour as the one of the twists along the Nakajima \( \mathbb{P} \)-functors that is described in Lemma 7.5.3. That means that we should have

\[
\begin{align*}
pt_{m-\ell}(\mathcal{C}(x)) & \cong \begin{cases} 
\mathcal{C}(x)[-2(m - \ell - 1)] & \text{for } x \in X^m_{\nu(m-\ell)}, \\
\mathcal{C}(x) & \text{for } x \in \bar{X}^m_{\nu(m-\ell)}
\end{cases} \\
t_{m-2,2}(\mathcal{C}(x)) & \cong \begin{cases} 
\mathcal{C}(x)[-1] & \text{for } x \in X^m_{\nu(2)}, \\
\mathcal{C}(x) & \text{for } x \in \bar{X}^m_{\nu(2)}.
\end{cases}
\end{align*}
\]

The main evidence is that there is in fact always an autoequivalence one can consider as \( t_{m-2,2} \), namely the composition \( \Phi^{-1} M_{\phi} \Phi \mathcal{O}(D/2) \) where \( D \) denotes the boundary divisor of the Hilbert scheme. Indeed, for \( x \in X^m_{\nu(2)} \) we have \( \Phi^{-1}(\mathcal{C}(x)) \cong \mathcal{O}_{m-1}(\pi(x))(-1) \) and \( \Phi^{-1}(\mathcal{C}(x) \otimes a) \cong \mathcal{O}_{m-1}(\pi(x))(-2)[1] \) where \( \pi: X^m \to S^mX \) is the quotient morphism and \( \mu: X^m \to S^mX \) is the Hilbert–Chow morphism. This is shown in the case \( m = 2 \) in [6, Proposition 4.4 & Remark 4.6] and the proof for general \( m \) is the same. Furthermore, \( \mathcal{O}(D/2)|_{m-1}(\pi(x)) \cong \mathcal{O}_{m-1}(\pi(x))(-1) \). Thus,

\[
\Phi^{-1} M_{\phi} \Phi \mathcal{O}(D/2)(\Phi^{-1}(\mathcal{C}(x)) \cong \Phi^{-1}(\mathcal{C}(x))[-1].
\]

As explained in Section 7.5.8 there are also autoequivalences of \( \mathcal{D}_C^b(C^m) \) that act as a shift on certain strata \( C^m_{\nu(k)} \) and as the identity on \( \mathcal{C}^m_{\nu(k)} \). Thus, one might also guess that this kind of autoequivalences exist in \( \mathcal{D}_C^b(X^m) \) for \( X \) a smooth variety of arbitrary dimension, even though for \( \dim X \geq 3 \) the functors \( H_{\ell,n} \) are far from being fully faithful or \( \mathbb{P} \)-functors; compare Remark 7.3.9.

References


210


Chapter 8

Symmetric quotient stacks and Heisenberg actions


Abstract

For every smooth projective variety $X$, we construct an action of the Heisenberg algebra on the direct sum of the Grothendieck groups of all the symmetric quotient stacks $[X^n/\mathfrak{S}_n]$ which contains the Fock space as a subrepresentation. The action is induced by functors on the level of the derived categories which form a weak categorification of the action.

8.1 Introduction

A celebrated theorem of Nakajima [Nak97] and Gromovski [Gro96] identifies the cohomology of Hilbert schemes of points on a surface with the Fock space representation of the Heisenberg Lie algebra associated to the cohomology of the surface itself.

One would like to lift the Heisenberg action to the level of the Grothendieck groups or, even better, the derived categories of the Hilbert schemes. Schiffmann and Vasserot [SV13] as well as Feigin and Tsymbaliuk [FT11] constructed a Heisenberg action on the equivariant Grothendieck groups in the case of the affine plane. In [CL12], Cautis and Licata constructed a categorical Heisenberg action on the derived categories of the Hilbert schemes in the case that the surface is a minimal resolution of a Kleinian singularity. Their construction makes use of the derived McKay correspondence between the derived category of the Hilbert scheme of points on a smooth quasi-projective surface and the derived category of the symmetric quotient stack associated to the surface; see [BKR01] and [Hai01].

In this paper, we generalise the construction of [CL12] to obtain functors between the derived categories of the symmetric quotient stacks of an arbitrary smooth complete variety which descend to a Heisenberg action on the Grothendieck groups of the symmetric quotient stacks. Our construction can also be seen as a global version of a construction of Khovavov [Kho14] which deals with the case that the variety is a point. Note, however, that there are deeper categorical structures in [CL12] as well as [Kho14] that we do not generalise; see also Section 8.3.5.
Our construction is much closer to the construction of [Gro96] (see also [Nak99, Ch. 9]) than to that of [Nak97]. In some sense, what our paper does is to fill in the proof of the claim made in [Gro96, footnote 3], though while working on the higher level of the derived categories.

8.1.1 Generators of the Heisenberg algebra

Let $V$ be a vector space over $\mathbb{Q}$ (not necessarily of finite dimension) with a symmetric bilinear form $\langle \cdot, \cdot \rangle$. Set $L_V = V \oplus \mathbb{Z}\{0\}$ and denote the image of $\beta \in V$ in the $n$-th factor of $L_V$ by $a_\beta(n)$ for $n \in \mathbb{Z} \setminus \{0\}$. The Heisenberg algebra $H_V$ associated to $(V, \langle \cdot, \cdot \rangle)$ is the unital $\mathbb{Q}$-algebra given by the tensor algebra $T(L_V) = \bigoplus_{i \geq 0} L_V^i$ modulo the relations

$$[a_\alpha(m), a_\beta(n)] = \delta_{m,-n} m \langle \alpha, \beta \rangle.$$  \tag{8.1}$$

Note that, with our definition, the Heisenberg algebra is the quotient of the universal enveloping algebra $U(h_V)$ of the Heisenberg Lie algebra associated to $(V, \langle \cdot, \cdot \rangle)$ where the central charge is identified with 1. Hence, every module over the Heisenberg algebra $H_V$ yields a representation of the Heisenberg Lie algebra $h_V$. In particular, the Fock space representation is given by a $H_V$-module; see Section 8.3.1.

We define elements $p^{(n)}_\beta$ and $q^{(n)}_\beta$ for $\beta \in V$ and $n$ a non-negative integer by the formulae

$$\sum_{n \geq 0} p^{(n)}_\beta z^n = \exp \left( \sum_{\ell \geq 1} \frac{a_\beta(-\ell)}{\ell} z^\ell \right), \quad \sum_{n \geq 0} q^{(n)}_\beta z^n = \exp \left( \sum_{\ell \geq 1} \frac{a_\beta(\ell)}{\ell} z^\ell \right).$$  \tag{8.2}$$

As $\beta$ runs through a basis of $V$ and $n$ through the positive integers, the elements $q^{(n)}_\beta$ and $p^{(n)}_\alpha$ together form a set of generators of $H_V$. In order to describe their relations, we introduce the following notation.

**Definition 8.1.1.** For $\chi \in \mathbb{Q}$ and $k$ a non-negative integer, we set

$$s^k \chi := \binom{\chi + k - 1}{k} := \frac{1}{k!}(\chi + k - 1)(\chi + k - 2)\cdots(\chi + 1)\chi.$$  

**Lemma 8.1.2.** The relations among the above generators are given by

$$[q^{(m)}_\alpha, q^{(n)}_\beta] = 0 = [p^{(m)}_\alpha, p^{(n)}_\beta],$$  \tag{8.3}$$

$$q^{(m)}_\alpha p^{(n)}_\beta = \sum_{k=0}^{\min\{m,n\}} s^k(\alpha, \beta) \cdot p^{(n-k)}_\beta q^{(m-k)}_\alpha.$$  \tag{8.4}$$

We will prove Lemma 8.1.2 in Section 8.2.1. Note the following simple but important fact.

**Lemma 8.1.3.** Let $W^*$ be a finite-dimensional graded vector space. Then the Euler characteristic of its symmetric product $S^k W^*$ (formed in the graded sense) is given by

$$\chi(S^k W^*) = s^k(\chi(W^*)).$$

The Euler characteristic of a graded vector space is defined as the alternating sum of its graded pieces: $\chi(W^*) := \sum_{i \in \mathbb{Z}} (-1)^i \dim V^i$.  

214
8.1.2 Construction and results

Let $X$ be a smooth complete variety over a field $k$ of characteristic zero. For a non-negative integer $\ell$, we consider the cartesian product $X^\ell$ together with the natural action of the symmetric group $S_\ell$ given by permuting the factors. The associated quotient stack $[X^\ell/\mathfrak{S}_\ell]$ is called the $\ell$-th symmetric quotient stack. Its derived category can equivalently be described as the $\mathfrak{S}_\ell$-equivariant derived category of the cartesian product, that means $D([X^\ell/\mathfrak{S}_\ell]) \cong D_{\mathfrak{S}_\ell}(X^\ell)$.

We set

$$\mathbb{D} := \bigoplus_{\ell \geq 0} D_{\mathfrak{S}_\ell}(X^\ell)$$

For $1 \leq n \leq N$ and $\beta \in D(X)$, we define the functor

$$P_{N,\beta}^{(n)}: D_{\mathfrak{S}_{N-n}}(X^{N-n}) \to D_{\mathfrak{S}_N}(X^N), \quad E \mapsto \text{Inf}_{\mathfrak{S}_{n}\times\mathfrak{S}_{N-n}}^E(\beta^{\otimes n} \boxtimes E). \quad (8.5)$$

Note that, for $E \in D_{\mathfrak{S}_{N-n}}(X^{N-n})$, we can consider $\beta^{\otimes n} \boxtimes E$ canonically as an object of $D_{\mathfrak{S}_{n}\times\mathfrak{S}_{N-n}}(X^N)$. The inflation functor $\text{Inf}_{\mathfrak{S}_{n}\times\mathfrak{S}_{N-n}}^E: D_{\mathfrak{S}_{n}\times\mathfrak{S}_{N-n}}(X^N) \to D_{\mathfrak{S}_N}(X^N)$ is the adjoint of the forgetful functor; see Section 8.2.2 for details on the functor $\text{Inf}$ and Section 8.2.4 for details on the functor $P_{N,\beta}^{(n)}$. We set

$$P_{\beta}^{(n)} := \bigoplus_{N \geq n} P_{N,\beta}^{(n)}: \mathbb{D} \to \mathbb{D} \quad \text{for } n \geq 1, \quad P_{\beta}^{(0)} := \text{id}: \mathbb{D} \to \mathbb{D}.$$

Finally, we define $Q_{\beta}^{(n)}: \mathbb{D} \to \mathbb{D}$ as the right adjoint of $P_{\beta}^{(n)}$.

**Theorem 8.1.4.** For every $\alpha, \beta \in D(X)$ and $n, m \in \mathbb{N}$, we have the relations

$$Q_{\alpha}^{(m)} Q_{\beta}^{(n)} \cong Q_{\beta}^{(n)} Q_{\alpha}^{(m)}, \quad P_{\alpha}^{(m)} P_{\beta}^{(n)} \cong P_{\beta}^{(n)} P_{\alpha}^{(m)} \quad (8.6)$$

$$Q_{\alpha}^{(m)} P_{\beta}^{(n)} \cong \bigoplus_{k=0}^{\min\{m,n\}} S^k \text{Hom}^*(\alpha, \beta) \boxtimes_k P_{\beta}^{(n-k)} Q_{\alpha}^{(m-k)} \quad (8.7).$$

We denote by $K([X^n/\mathfrak{S}_n]) \cong K_{\mathfrak{S}_n}(X^n)$ the equivariant Grothendieck group with coefficients in $\mathbb{Q}$ and set $\mathbb{K} := \bigoplus_{p \geq 0} K_{\mathfrak{S}_p}(X^n) = K(\mathbb{D})$. We consider $K(X)$ together with the bilinear form $\langle \_, \_ \rangle$ given by the Mukai pairing, i.e.

$$\langle [\alpha], [\beta] \rangle := \chi(\alpha, \beta) := \chi(\text{Hom}^*(\alpha, \beta)) \quad (8.8)$$

where $[\alpha]$ denotes the class of an object $\alpha \in D(X)$ in the Grothendieck group $K(X)$.

**Corollary 8.1.5.** The descent of the functors $P_{\beta}^{(n)}$ and $Q_{\beta}^{(n)}$ to the level of the Grothendieck groups makes $\mathbb{K}$ into a representation of the Heisenberg algebra $H_{K(X)}$.

**Proof.** Let $p_{\beta}^{(n)} := [P_{\beta}^{(n)}]: \mathbb{K} \to \mathbb{K}$ and $q_{\beta}^{(n)} := [Q_{\beta}^{(n)}]: \mathbb{K} \to \mathbb{K}$ be the induced maps on the level of the Grothendieck groups. By Lemma 8.1.3 and (8.8), the relations of Theorem 8.1.4 translate to the $p_{\beta}^{(n)}$ and $q_{\beta}^{(n)}$ satisfying the relations of Lemma 8.1.2. \[\square\]
The category $\mathcal{D}$ together with the functors $Q^{(n)}_\beta$ and $P^{(n)}_\beta$ can be regarded as a categorification of the Heisenberg representation $\mathbb{K}$.

Since all the $q^{(n)}_\beta$ vanish on $\mathbb{K}_0 = \mathbb{K}_{\mathfrak{h}}(X^0) = \mathbb{K}(\text{point}) \cong \mathbb{Q}$, one gets an embedding of the Fock space representation into $\mathbb{K}$; see Section 8.3.1 for details.

We will give the proof of Theorem 8.1.4 in Sections 8.2.5 and 8.2.6. One main ingredient is the following easy fact.

**Lemma 8.1.6 ("Symmetric Künneth formula").** Let $\alpha, \beta \in \mathcal{D}(X)$ and $k \in \mathbb{N}$. Then

$$\text{Hom}^*_\mathcal{D}_{\mathfrak{h}}(X^k)(\beta^{\otimes k}, \alpha^{\otimes k}) \cong S^k \text{Hom}^*(\beta, \alpha)$$

### 8.1.3 Conclusion

The reason for the occurrence of Heisenberg actions in the context of symmetric quotient stacks, and hence also of Hilbert schemes of points on surfaces, can be summarised in the following simple and, in the author’s opinion, satisfying way:

1. The Heisenberg algebra has generators whose relations involve the numbers $s^n\langle \alpha, \beta \rangle$ as coefficients; see Lemma 8.1.2.

2. A “vector-spaceification” of these numbers is given by the graded vector spaces $S^n \text{Hom}^*(\alpha, \beta)$; see Lemma 8.1.3.

3. Because of the symmetric Künneth formula (Lemma 8.1.6), these graded vector spaces show up very naturally in the context of derived categories of symmetric quotient stacks. In particular, the relatively simple construction (8.5) yields a categorical action of the Heisenberg algebra.

### 8.1.4 Organisation of the paper

In Section 8.2.1, we prove Lemma 8.1.2. Afterwards, we recall some basic facts on equivariant derived categories and functors in Section 8.2.2, introduce some notation in Section 8.2.3, and describe the functors $P^{(n)}_\beta$ and $Q^{(n)}_\beta$ in some more detail in Section 8.2.4. In Sections 8.2.5 and 8.2.6, we give the proof of Theorem 8.1.4.

We explain in Section 8.3.1 the embedding of the Fock space representation into $\mathbb{K}$. In Section 8.3.3, we define further functors which give a categorification of the action on $\mathbb{K}$ with respect to the transposed generators of $H_{\mathbb{K}(X)}$. In particular, this recovers the construction of [Kho14]. In Section 8.3.4 we discuss some generalisations and variants, for example to the non-complete case and the case where we replace the category $\mathcal{D}(X)$ by some equivariant category. Finally, we point out two open problems in Section 8.3.5.

**Conventions** With the exception of Section 8.3.4, $X$ will be a smooth complete variety over a field $k$ of characteristic zero. Its derived category is understood to be the bounded derived category of coherent sheaves, i.e. $\mathcal{D}(X) := \mathcal{D}^b(\text{Coh}(X))$. All functors between derived categories are assumed to be derived although this will not be reflected in the notation. By convention, $X^0 := \text{pt}$ is a point.

**Acknowledgements.** The author was financially supported by the research grant KR 4541/1-1 of the DFG. He thanks Sabin Cautis, Daniel Huybrechts, Ciaran Meachan, David Ploog, Miles Reid, Pawel Sosna, and the referee for helpful comments.
8.2 Proofs

8.2.1 Proof of Lemma 8.1.2

Let $\alpha, \beta \in V$ and $m, n \in \mathbb{N}$. The vanishing of the commutators $[q^{(m)}_\alpha, q^{(n)}_\beta]$ and $[p^{(m)}_\alpha, p^{(n)}_\beta]$ follows immediately from the fact that $[a_{\alpha}(r), a_{\beta}(s)] = 0$ if $r$ and $s$ are both positive or both negative.

For the proof of relation (8.4), we follow closely the proof of [CL12, Lem. 1]. Set

$$A(z) := \sum_{\ell \geq 1} \frac{a_\beta(-\ell)}{\ell} z^\ell , \quad B(w) := \sum_{\ell \geq 1} \frac{a_\alpha(\ell)}{\ell} w^\ell$$

so that $q^{(m)}_\alpha p^{(n)}_\beta = [w^m z^n] \exp(B) \exp(A)$. We also set $\chi := \langle \alpha, \beta \rangle$. By relation (8.1), we have

$$[B, A] = \sum_{\ell \geq 1} \frac{\ell \cdot \chi}{\ell^2} w^\ell z^\ell = \chi \sum_{\ell \geq 1} \frac{(wz)^\ell}{\ell} = -\chi \log(1 - wz) . \quad (8.9)$$

In particular, $[B, A]$ commutes with $A$ and $B$. Thus,

$$\exp(B) \exp(A) = (1 - wz)^{-\chi} \exp(A) \exp(B) ; \quad (8.10)$$

see [Nak99, Lem. 9.43]. Recall the binomial formula

$$(1 - t)^{-\chi} = \sum_{k \geq 0} s^k \chi \cdot t^k . \quad (8.11)$$

Together, (8.9), (8.10), and (8.11) give

$$\exp(B) \exp(A) = (1 - wz)^{-\chi} \exp(A) \exp(B) = \left( \sum_{k \geq 0} s^k \chi \cdot (wz)^k \right) \exp(A) \exp(B) .$$

Comparing the coefficients of $w^m z^n$ on both sides gives relation (8.4).

In order to show that there are no further relations among the generators $q^{(n)}_\beta$ and $p^{(n)}_\beta$ we introduce the following notation. Let $\{\beta_i\}_{i \in I}$ be a basis of $V$. Consider maps

$$\nu: I \to \prod_{n \geq 0} \\{ \text{partitions of } n \} \quad (8.12)$$

such that $\nu(i) \neq 0$ only for a finite number of $i \in I$. We write the partitions in the form $\nu(i) = 1^{\nu(i)_1} 2^{\nu(i)_2} \ldots$. We fix a total order on $I$ and set

$$a(\nu) := \prod_{i \in I} \prod_{k \geq 1} a_{\beta_i}(k)^{\nu(i)_k} , \quad a(-\nu) := \prod_{i \in I} \prod_{k \geq 1} a_{\beta_i}(-k)^{\nu(i)_k} ,$$

$$q(\nu) := \prod_{i \in I} \prod_{k \geq 1} (q^{(k)}_{\beta_i})^{\nu(i)_k} , \quad p(\nu) := \prod_{i \in I} \prod_{k \geq 1} (p^{(k)}_{\beta_i})^{\nu(i)_k} .$$

Here, the inner product is formed with respect to the usual order of $\mathbb{N}$ and the outer product with respect to the fixed order of $I$. Since the only relation between the generators $a_{\beta}(k)$ is the commutator relation (8.1), the elements $a(-\nu)a(\mu)$ form a basis of $\mathbb{H}_V$ as a vector space.
In the algebra with generators $q^{(n)}_{\beta_i}$ and $p^{(n)}_{\beta_i}$ and the relations (8.3) and (8.4), one can write every element as a linear combination of the form

$$\sum_{\nu, \mu} \lambda(\nu, \mu)p(\nu)q(\mu), \quad \lambda(\mu, \nu) \in \mathbb{Q}. \quad (8.13)$$

Thus, it is sufficient to show that the $p(\nu)q(\mu)$ are linearly independent in $H_V$. For two maps $\nu, \nu'$ as in (8.12) we say that $\nu$ is coarser than $\nu'$, and write $\nu \succ \nu'$, if $|\nu(i)| = |\nu'(i)|$ and $\nu(i)$ is a coarser partition than $\nu'(i)$ for every $i \in I$. For pairs, we set $(\nu, \mu) \succ (\nu', \mu')$ if $\nu \succ \nu'$ and $\mu \succ \mu'$. Note that by (8.2), we have

$$p^{(n)}_{\beta} = a_\beta(-n) + \text{(linear combination of } a_\beta(-1)^{\nu_1} a_\beta(-2)^{\nu_2} \cdots \text{ for partitions } \nu \text{ of } n)$$

and similarly for $q^{(n)}_{\beta}$. Thus,

$$p(\nu)q(\mu) = a(-\nu)a(\mu) + \text{(linear combination of } a(-\nu')a(\mu') \text{ for } (\nu, \mu) \succ (\nu', \mu')). \quad (8.14)$$

Let $x = \sum_{\nu, \mu} \lambda(\nu, \nu)p(\nu)q(\mu)$ as in (8.13) such that not all coefficients vanish. Pick a pair $(\nu_0, \mu_0)$ with $\lambda(\nu_0, \mu_0) \neq 0$ such that for each coarser pair the corresponding coefficient vanishes. Then, by (8.14), $x$ is a linear combination of the $a(-\nu)b(-\mu)$ such that the coefficient of $a(-\nu_0)b(\mu_0)$ equals $\lambda(\nu_0, \mu_0)$. Since the $a(-\nu)b(\mu)$ form a basis, $x \neq 0$ in $H_V$.

### 8.2.2 Basic facts about equivariant functors

For details on equivariant derived categories and functors between them, we refer to [BKR01, Sect. 4]. Let $G$ be a finite group acting on a smooth projective variety $M$. The \textit{equivariant derived category} $D_G(M)$ has as objects pairs $(E, \lambda)$ where $E \in D(M)$ and $\lambda$ is a $G$-linearisation of $E$. A $G$-linearisation is a family of isomorphisms $(E \xrightarrow{\simeq} g^*E)_{g \in G}$ in $D(M)$ such that for every pair $g, h \in G$ the composition $E \xrightarrow{\lambda_{gh}} g^*h^*E \simeq (hg)^*E$ equals $\lambda_{gh}$. The morphisms are morphisms in the ordinary derived category $D(X)$ which are compatible with the linearisations. The category $D_G(M)$ is equivalent to the bounded derived category $D^b(\text{Coh}_G(M))$ of $G$-equivariant coherent sheaves on $M$; see [Che14] or [Ela14].

For a subgroup $U \subseteq G$, there is the functor $\text{Res}^G_U: D_G(M) \to D_U(M)$ given by restricting the linearisations. It has the \textit{inflation} (also known as \textit{induction}) functor $\text{Inf}^G_U: D_U(M) \to D_G(M)$ as a left and right adjoint. Let $U \setminus G$ be the left cosets and $E \in D(X)$. Then

$$\text{Inf}^G_U(E) \cong \bigoplus_{\sigma \in U \setminus G} \sigma^*E \quad (8.15)$$

with the $G$-linearisation of $\text{Inf}^G_U(E)$ given by a combination of the $U$-linearisation of $E$ and permutation of the direct summands.

Let $G$ act trivially on $M$. Then there is the functor $\text{triv}: D(M) \to D_G(M)$ which equips every object with the trivial linearisation. It has the functor of invariants $(\underline{-})^G: D_G(M) \to D(M)$ as a left and right adjoint.

We use the following simple principle for the computation of invariants; see e.g. [7, Sect. 3.5]. Let $\mathcal{E} = (E, \lambda)$ be a $G$-equivariant sheaf such that $E = \oplus_{i \in \mathcal{K}} E_i$ for some finite index set $\mathcal{K}$. Assume that there is an action of $G$ on $\mathcal{K}$ such that $\lambda_g(E_i) = E_{g_i}$ for all $i \in \mathcal{K}$ and $g \in G$. We say that the linearisation $\lambda$ induces the action on the index set $\mathcal{K}$.
Lemma 8.2.1. Let $\mathcal{R} \subset \mathcal{K}$ be a set of representatives of the $G$-orbits in $\mathcal{K}$ under the action induced by the linearisation. Then $E^G \cong \bigoplus_{i \in \mathcal{R}} E_i^G$, where $G_i \subset G$ is the stabiliser subgroup.

8.2.3 Combinatorial notations

For a finite set $I$ we set $X^I := \prod_I X \cong X^{|I|}$. For $J \subset I$ we write $\text{pr}_J^I : X^I \to X^J$ for the projection. We often drop the index $I$ in the notation and simply write $\text{pr}_J$ when the source of the projection should be clear from the context.

For $n \in \mathbb{N}$, we set $[n] := \{1, n\} = \{1, 2, \ldots, n\}$. By convention, $0 := \emptyset$. Later, there will be a fixed number $N \in \mathbb{N}$ such that all occurring finite sets will be subsets of $N$. Hence, we will often write $N$ instead of $N$ to ease the notation. Furthermore, for a subset $J \subset N$ we set $J := N \setminus J$. For $0 \leq n \leq N$ we set $\pi := [n] = N \setminus [n] = [n+1, N]$.

8.2.4 The functors $P_{\beta}^{(n)}$ and $Q_{\alpha}^{(n)}$

For every object $\beta \in D(X)$ and $N \geq n$, the functor $P_{N,\beta}^{(n)} : D_{\mathfrak{S}_{N-n}}(X^{N-n}) \to D_{\mathfrak{S}_N}(X^N)$ is given by the composition

$$D_{\mathfrak{S}_{N-n}}(X^{N-n}) \xrightarrow{\text{triv}} D_{\mathfrak{S}_n \times \mathfrak{S}_{N-n}}(X^{N-n}) \xrightarrow{\text{pr}_n^*} D_{\mathfrak{S}_n \times \mathfrak{S}_{N-n}}(X^N)$$

$$\otimes \text{pr}_1^{[n]} \beta^{\otimes n} \xrightarrow{\otimes \text{pr}_n^*[\mathfrak{S}_{[n]}]} D_{\mathfrak{S}_n \times \mathfrak{S}_{N-n}}(X^N) \xrightarrow{\text{Res}_{\mathfrak{S}_{[n]} \times \mathfrak{S}_n}} D_{\mathfrak{S}_n}(X^N),$$

where we use the identifications $X^{[n]} \cong X^n$ and $X\pi = X^{[n+1,N]} \cong X^{N-n}$. Hence, the right-adjoint $Q_{N,\beta}^{(n)} : D_{\mathfrak{S}_N}(X^N) \to D_{\mathfrak{S}_{N-n}}(X^{N-n})$ is the composition

$$D_{\mathfrak{S}_{N-n}}(X^{N-n}) \xleftarrow{\otimes \text{pr}_n^*[\mathfrak{S}_{[n]}]} D_{\mathfrak{S}_n \times \mathfrak{S}_{N-n}}(X^{N-n}) \xrightarrow{\text{pr}_n^*} D_{\mathfrak{S}_n \times \mathfrak{S}_{N-n}}(X^N)$$

$$\otimes \text{pr}_1^{[n]} \beta^{\otimes n} \xleftarrow{\otimes \text{pr}_n^*[\mathfrak{S}_{[n]} \times \mathfrak{S}_n]} D_{\mathfrak{S}_n \times \mathfrak{S}_{N-n}}(X^N) \xrightarrow{\text{Res}_{\mathfrak{S}_{[n]} \times \mathfrak{S}_n}} D_{\mathfrak{S}_n}(X^N).$$

Note that there is a canonical isomorphism $(\beta^\vee)^{\otimes n} \cong (\beta^{\otimes n})^\vee$ so that we simply write $\beta^{\otimes n}$ without ambiguity. Since

$$\sigma^*(\text{pr}_1^{[n]} \beta^{\otimes n} \otimes \text{pr}_n^* E) \cong \text{pr}_{\sigma^{-1}(\mathfrak{S})}^{\otimes n} \otimes \text{pr}_{\sigma^{-1}(\mathfrak{S})}^* E$$

for every $E \in D_{\mathfrak{S}_{N-n}}(X^{N-n})$ and $\sigma \in \mathfrak{S}_n$, we get by (8.15)

$$P_{N,\beta}^{(n)}(E) \cong \bigoplus_{I \in \mathcal{K}, |I| = n} \text{pr}_I^* \beta^{\otimes n} \otimes \text{pr}_n^* E$$

(8.18)

Furthermore, the functor $Q_{N,\alpha}^{(n)}$ is given on objects $F \in D_{\mathfrak{S}_N}(X^N)$ by

$$Q_{N,\alpha}^{(m)}(F) \cong \text{pr}_{\mathfrak{S}_m}^* (\text{pr}_m^* \alpha^{\otimes m} \otimes F)$$

(8.19)

Remark 8.2.2. Let $I$ be any set of cardinality $N$ and $J \subset I$ be any subset of cardinality $m$. Making the identifications $X^I \cong X^N$, $X^J \cong X^m$, and $X^{I \setminus J} \cong X^{N-m}$ we can also write

$$Q_{N,\alpha}^{(m)}(F) \cong \text{pr}_{I \setminus J}^* (\text{pr}_J^* \alpha^{\otimes m} \otimes F) \mathfrak{S}_J.$$
8.2.5 Proof of relation (8.6)

In order to verify the relations (8.6) of Theorem 8.1.4 it is sufficient to show the first relation, i.e. $Q^{(m)}_{\alpha}Q^{(n)}_{\beta} \cong Q^{(n)}_{\beta}Q^{(m)}_{\alpha}$ for $\alpha, \beta \in D(X)$ and $m, n \in \mathbb{N}$. The relation $P^{(m)}_{\alpha}P^{(n)}_{\beta} \cong P^{(n)}_{\beta}P^{(m)}_{\alpha}$ follows by adjunction. This means that we have to show that

$$ Q^{(m)}_{N-n,\alpha}Q^{(n)}_{N,\beta} \cong Q^{(n)}_{N-m,\beta}Q^{(m)}_{N,\alpha} : D_{\mathbb{N}}(X^N) \to D_{\mathbb{N}}(X^{N-m-n}) \quad \text{for every } N \geq n + m. $$

Indeed, using Remark 8.2.2, the projection formula along $pr^n_{\pi}$, the projection formula along $pr^{N}_{[n+1,n+m]}$, and finally Remark 8.2.2 again, we get isomorphisms

$$ Q^{(m)}_{N-n,\alpha}Q^{(n)}_{N,\beta}(F) \cong \left[pr^n_{\pi,m} \left(\left(pr^n_{\pi,n+1,n+m}\alpha^{\vee \otimes m} \otimes \left[pr^n_{\pi,n} (pr^n_{\pi,n} \beta^{\vee \otimes n} \otimes F)\right]_{\mathbb{N}}\right)_{\mathbb{N}+1,n+m\mathbb{N}} \right) \right]_{\mathbb{N}+1,n+m\mathbb{N}} \cong \left[pr^n_{\pi,n+1,n+m} (pr^n_{\pi,n} \beta^{\vee \otimes n} \otimes \left[pr^n_{\pi,n} (pr^n_{\pi,n} \alpha^{\vee \otimes n} \otimes F)\right]_{\mathbb{N}}\right)_{\mathbb{N}+1,n+m\mathbb{N}} \right]_{\mathbb{N}} \cong Q^{(n)}_{N-m,\beta}Q^{(m)}_{N,\alpha}(F) $$

which are functorial in $F \in D_{\mathbb{N}}(X^N)$.

8.2.6 Proof of relation (8.7)

Let $N \geq \max\{m, n\}$ and $E \in D_{\mathbb{N}}(X^{N-n})$. Combining (8.18) and (8.19) we get

$$ Q^{(m)}_{\alpha}P^{(n)}_{\beta}(E) \cong \bigoplus_{I \in \mathcal{K}} T_I^{\mathfrak{S}_m \times \mathfrak{S}_{[k+1,m]}} $$

where

$$ \mathcal{K} = \{ I \subset \mathbb{N}, |I| = n \} \quad \text{and} \quad T_I = pr^{\pi}_{m} \left( pr^{\pi}_{m,n} \alpha^{\vee \otimes m} \otimes pr^{\pi}_{J} \beta^{\vee \otimes n} \otimes pr^{\pi}_{\ell} E \right). $$

The $\mathfrak{S}_m$-linearisation of $\otimes_{I \in \mathcal{K}} T_I$ induces on the index set $\mathcal{K}$ the action $\sigma \cdot I = \sigma(I)$. A set of representatives of the $\mathfrak{S}_m$-orbits in $\mathcal{K}$ is given by

$$ \mathcal{R} = \{ I = k \cup J \mid n + m - N \leq k \leq \min\{m, n\}, J \subset m, |J| = n - k \}. $$

Thus, Lemma 8.2.1 yields

$$ Q^{(m)}_{\alpha}P^{(n)}_{\beta}(E) \cong \bigoplus_{I \in \mathcal{K}} T_I^{\mathfrak{S}_m \times \mathfrak{S}_{[k+1,m]}}. \quad (8.20) $$

Let $I = k \cup J \in \mathcal{R}$ and consider the diagram

$$ \begin{array}{c}
X^{m \cap J} \xrightarrow{pr^{\pi}_{m \setminus J}} X^m \xrightarrow{pr^{\pi}_{J}} X^{m \setminus J} = X^J \\
X^{m \cap J} \xrightarrow{pr^{\pi}_{m \setminus J}} X^m \xrightarrow{pr^{\pi}_{J}} X^{m \setminus J} = X^J \\
X^{N} \xrightarrow{pr^{\pi}_{k}} X^{m \cap J} \xrightarrow{pr^{\pi}_{k}} X^k \\
X^{N} \xrightarrow{pr^{\pi}_{J}} X^{m \cap J} \xrightarrow{pr^{\pi}_{J}} X^k \\
X^{N} \xrightarrow{pr^{\pi}_{[k+1,m]}} X^{m \cap J} \xrightarrow{pr^{\pi}_{[k+1,m]}} X^{[k+1,m]}.
\end{array} $$

(8.21)
where

\[ T_I \cong \text{pr}_m \text{pr}_N \left( [\text{pr}_m \beta^{\mathbb{Z} - k} \otimes \text{pr}_N E \otimes \text{pr}_N \alpha^{\mathbb{Z} - k}] \right) \otimes \text{pr}_N^N (\alpha \otimes \beta)^{\mathbb{Z} - k} \].

By the projection formula along \( \text{pr}_N \), we get

\[ T_I \cong \text{pr}_m \left( [\text{pr}_m \beta^{\mathbb{Z} - k} \otimes \text{pr}_N E \otimes \text{pr}_N \alpha^{\mathbb{Z} - k}] \right) \otimes \text{pr}_N^N (\alpha \otimes \beta)^{\mathbb{Z} - k} \].

By the fact that \( X^N = X^k \times X^k \) and Künneth formula

\[ \text{pr}_N^N (\alpha \otimes \beta)^{\mathbb{Z} - k} \cong \mathcal{O}_{X^k} \otimes_k \text{Hom}^* (\alpha^{\mathbb{Z} - k}, \beta^{\mathbb{Z} - k}) \cong \mathcal{O}_{X^k} \otimes_k \text{Hom}^* (\alpha, \beta)^{\mathbb{Z} - k} \].

This gives

\[ T_I \cong \text{Hom}^* (\alpha, \beta)^{\mathbb{Z} - k} \otimes_k \widehat{T}_I \quad , \quad \widehat{T}_I = \text{pr}_m \left( [\text{pr}_m \beta^{\mathbb{Z} - k} \otimes \text{pr}_N E \otimes \text{pr}_N \alpha^{\mathbb{Z} - k}] \right) \otimes \text{pr}_N^N (\alpha \otimes \beta)^{\mathbb{Z} - k} \].

By the projection formula along \( \text{pr}_m \) and base change along the cartesian square in (8.21),

\[ \widehat{T}_I \cong \text{pr}_m \beta^{\mathbb{Z} - k} \otimes \text{pr}_N E \otimes \text{pr}_N \alpha^{\mathbb{Z} - k} \]

\[ \cong \text{pr}_m \beta^{\mathbb{Z} - k} \otimes \text{pr}_N E \otimes \text{pr}_N \alpha^{\mathbb{Z} - k} \]

In summary,

\[ T_I \cong \text{Hom}^* (\alpha, \beta)^{\mathbb{Z} - k} \otimes_k \text{pr}_m \beta^{\mathbb{Z} - k} \otimes \text{pr}_m \beta^{\mathbb{Z} - k} \otimes \text{pr}_N \alpha^{\mathbb{Z} - k} \]

Taking \( \mathcal{G}_k \times \mathcal{G}_{[k+1,m]} \)-invariants yields

\[ T_I^{\mathcal{G}_k \times \mathcal{G}_{[k+1,m]}} \cong S^k \text{Hom}^* (\alpha, \beta) \otimes \text{pr}_m \beta^{\mathbb{Z} - k} \otimes \text{pr}_m \beta^{\mathbb{Z} - k} \otimes \text{pr}_N \alpha^{\mathbb{Z} - k} \]

Plugging this into (8.20) gives

\[ Q^{(m)}_\alpha P^{(n)}_\beta (E) \cong \bigoplus_{k=n+m-N} S^k \text{Hom}^* (\alpha, \beta) \otimes \widehat{T}_k \]

where

\[ \widehat{T}_k := \bigoplus_{J \subseteq m, |J| = n-k} \text{pr}_m \beta^{\mathbb{Z} - k} \otimes \text{pr}_m \beta^{\mathbb{Z} - k} \otimes \text{pr}_N \alpha^{\mathbb{Z} - k} \]

Using Remark 8.2.2, we see that \( \widehat{T}_k \cong P^{(n-k)}_{N-m, \beta} Q^{(m-k)}_{N-n, \alpha} (E) \). Hence,

\[ Q^{(m)}_\alpha P^{(n)}_\beta (E) \cong \bigoplus_{k=n+m-N} S^k \text{Hom}^* (\alpha, \beta) \otimes \bigoplus_{J \subseteq m, |J| = n-k} P^{(n-k)}_{N-m, \beta} Q^{(m-k)}_{N-n, \alpha} (E) \]

Since all the isomorphisms used above are functorial, this gives the isomorphism (8.7).
8.3 Further remarks

8.3.1 The Fock space

The Fock space representation of the Heisenberg algebra is defined as the quotient $F_V := H_V/I$ where $I$ is the left ideal generated by all the $a_\beta(n)$ with $n > 0$. Let $1 \in F_V$ be the class of $1 \in H_V$. The Fock space is an irreducible representation. Hence, given any representation $M$ of $H_V$ together with an element $0 \neq x \in M$ such that $a_\beta(n) \cdot x = 0$ for all $\beta \in V$ and $n > 0$, the map

$$F_V \to M, \quad 1 \mapsto x$$

gives an embedding of $H_V$-representations.

Let $\mathcal{K} = \bigoplus_{\ell \geq 0} K_{\ell}(X^\ell)$ be the representation described in Section 8.1.2. Then every non-zero $x \in K_{\ell}(X^0) \cong K(\text{point}) \cong \mathbb{Q}$ is annihilated by all the $d_\beta^{(n)}$ and hence by the $a_\beta(n)$ for $n > 0$. Thus, the Fock space can be realised as a subrepresentation of $\mathcal{K}$.

In general, it is a proper subrepresentation, i.e. $F_{K(X)} \subseteq \mathcal{K}$. However, if $K(X)$ is of finite dimension and the exterior product $K(X)^{\otimes n} \to K(X^n)$ is an isomorphism, for example if $X$ has a cellular decomposition, we do have the equality $F_{K(X)} = \mathcal{K}$. In this case,

$$\mathcal{K}_\ell := K_{\ell}(X^\ell) \cong \bigoplus_{\nu \text{ partition of } \ell} (S^{\nu_1} K(X)) \otimes (S^{\nu_2} K(X)) \otimes \ldots$$

as follows from the general decomposition of equivariant K-theory described in [Vis91]. Thus, the dimensions of $F_{K(X)}$ and $\mathcal{K}$ agree in every degree $\ell$.

8.3.2 Left adjoints

The choice to define the functors $Q$ as the right (instead of left) adjoints of the functors $P$ is quite arbitrary. Indeed, let $\tilde{Q}_\beta^{(n)}$ be the left adjoint of $P_\beta^{(n)}$. The only difference between $Q_{N,\beta}^{(n)}$ and $\tilde{Q}_{N,\beta}^{(n)}$ is that, for the latter, one has to replace $\text{pr}_{\pi_*}$ by $\text{pr}_{\pi!} = \text{pr}_{\pi!*}(\omega_0 \otimes \text{pr}_*^n \omega X)_{n \cdot \text{dim} X}$ in the composition (8.16). In particular, if $X$ is Calabi–Yau of even dimension, $Q_{N,\beta}^{(n)}$ and $\tilde{Q}_{N,\beta}^{(n)}$ only differ by a shift.

Taking the left-adjoints of both sides of (8.7), we get

$$\tilde{Q}_\alpha^{(m)} P_\beta^{(n)} \cong \bigoplus_{k=0}^{\min\{m,n\}} S^k \text{Hom}^*(\beta,\alpha)^\vee \otimes_k P_\beta^{(n-k)} \tilde{Q}_\alpha^{(m-k)}.$$

Since $\chi(_,\_)$ is symmetric, this relation also descends to (8.4) on the level of the Grothendieck group. Hence, the functors $Q$ and $P$ still categorify the Heisenberg action.

8.3.3 Transposed generators

There is yet another set of generators of the Heisenberg algebra, namely $p_\beta^{(1)}$, $q_\beta^{(1)}$ defined by

$$\sum_{n \geq 0} (-1)^n p_\beta^{(1)} z^n = \exp \left( - \sum_{\ell \geq 1} \frac{a_\beta(-\ell)}{\ell} z^\ell \right), \quad \sum_{n \geq 0} (-1)^n q_\beta^{(1)} z^n = \exp \left( - \sum_{\ell \geq 1} \frac{a_\beta(\ell)}{\ell} z^\ell \right);$$
compare [CL12, Sect. 2.2.2]. The relations among these generators are exactly the same as those between the $p^\alpha_\beta$ and $q^\alpha_\beta$. Moreover, one can mix these two sets of generators to get the set of generators consisting of $p^\alpha_\beta$ and $q^\alpha_\beta$ (or, alternatively, $p^{(1)}_\beta$ and $q^{(1)}_\beta$). These are the generators used by Khovanov in [Kho14] for his construction of a categorification of the Heisenberg algebra $H_V$ in the case that $V = Q$. The relations between these generators are

$$[q^\alpha_\beta, q^{(1)}_\beta] = 0 = [p^\alpha_\beta, p^{(1)}_\beta], \quad q^\alpha_\beta p^{(1)}_\beta = \sum_{k=0}^{\min\{m,n\}} (-1)^k s^k (-\langle \alpha, \beta \rangle) \cdot p^{(n-k)}_\beta q^{(1-m-k)}_\beta.$$  

For $\beta \in D(X)$ and $n > 0$, we set $P_{\beta}^{(1)} := \bigoplus_{N \geq n} P^{(1)}_{N,\alpha}$ with

$$P^{(1)}_{N,\alpha} : D_E N(X_{N-n}) \to D_E N(X_N), \quad E \mapsto \text{Inf}_{E \times E}^N((\beta^{\otimes N} \otimes a_n) \boxtimes E).$$

Here, $a_n$ denotes the non-trivial character of $E_n$. Again, $Q^{(1)}_\beta$ is defined as the right-adjoint of $P_{\beta}^{(1)}$. Using the “anti-symmetric Künneth formula”

$$\text{Hom}^*_E(X^k)((\beta^{\otimes k} \otimes a_k, \alpha^{\otimes k}) \cong \wedge^k \text{Hom}^*(\beta, \alpha),$$

one can prove in complete analogy to Sections 8.2.5 and 8.2.6 the relations

$$Q^{(1)}_\alpha P^{(1)}_{\beta} \cong q^{(1)}_\beta q^{(1)}_\alpha, \quad P^{(m)}_{\alpha} P^{(n)}_{\beta} := P^{(n)}_{\beta} P^{(m)}_{\alpha}$$  \hspace{1cm} \text{(8.22)}

$$Q^{(1)}_\alpha P^{(1)}_{\beta} \cong \bigoplus_{k=0}^{\min\{m,n\}} \wedge^k \text{Hom}^*(\alpha, \beta) \boxtimes_k P^{(n-k)}_{\beta} Q^{(1-m-k)}_\alpha. \hspace{1cm} \text{(8.23)}$$

One can reformulate Lemma 8.1.3 as the formula $\chi(\wedge^k W^*) = (-1)^k s^k (-\chi(W^*))$ for $W^*$ a graded vector space. Thus, relations (8.22) and (8.23) show that the $P_{\beta}^{(1)}$ and $Q^{(1)}_\beta$ give again a categorical Heisenberg action, this time lifting the generators $p^{(1)}_\beta$ and $q^{(1)}_\beta$. The special case that $X$ is a point is exactly the construction of [Kho14].

### 8.3.4 Generalisations and variants

Since the construction (8.5) of the functors $P_{\beta}^{(n)}$ is very simple and formal, it generalises well to other settings beside smooth complete varieties.

Let $X$ be a non-complete smooth variety. Then Theorem 8.1.4 continues to hold if we assume that the objects $\alpha$ and $\beta$ have complete support, i.e. $\alpha, \beta \in D^c(X)$, which ensures that $\text{Hom}^*(\alpha, \beta)$ is a finite dimensional graded vector space. Thus, we get an induced action of the Heisenberg algebra associated to $K^c(X) := K(D^c(X))$ on $\mathbb{K}$ (note that $\mathbb{K}$ needs not necessarily be replaced by Grothendieck groups with finite support).

Similarly, if one wants to also drop the smoothness assumption one needs $\alpha$ and $\beta$ to be perfect objects with complete support.

Furthermore, one can replace the variety $X$ by a Deligne–Mumford stack $X$. The case that $X = [\text{pt}/G]$ for $G$ a finite group, gives a Heisenberg action on the representation rings of the wreath products, as $\bigoplus_{i \geq 0} K_{i}(\mathcal{X}^\ell) \cong \bigoplus_{i \geq 0} R(G \wr S_i)q_i$.  

223
8.3.5 Open problems

In [GK14], the symmetric product $S^n \mathcal{T}$ of a category $\mathcal{T}$ is defined in such a way that $S^n(D(X)) \cong D_{S^n}(X^n)$ for a smooth projective variety $X$. Thus, one may hope that our construction can be generalised to a setting where $D(X)$ is replaced by any (Hom-finite and symmetric monoidal) dg-enhanced triangulated category $\mathcal{T}$.

Note that in [Kho14] and [CL12] a categorification of the Heisenberg action is given in a much stronger sense than in this paper. In particular, 2-categories whose Grothendieck groups are isomorphic to the Heisenberg algebra are constructed. Clearly, one would like to do the same in our more general setting too.

References


Chapter 9

Remarks on the derived McKay correspondence for Hilbert schemes of points and tautological bundles

(arXiv:1612.04348.)

Abstract

We study the images of tautological bundles on Hilbert schemes of points on surfaces and their wedge powers under the derived McKay correspondence. The main observation of the paper is that using a derived equivalence differing slightly from the standard one considerably simplifies both the results and their proofs. As an application, we obtain shorter proofs for known results as well as new formulae for homological invariants of tautological sheaves. In particular, we compute the extension groups between wedge powers of tautological bundles associated to line bundles on the surface.

9.1 Introduction

Let \( G \) be a finite group which acts on a smooth variety \( M \). The McKay correspondence is a principle describing the relationship between the geometry of certain resolutions of the singularities of the quotient \( M/G \) and the representation theory of \( G \). Probably the most important example of the McKay correspondence in higher dimensions is the case where \( M = X^n \) is a power of a smooth surface with the symmetric group \( G = \mathfrak{S}_n \) permuting the factors. In this case, a crepant resolution of the quotient singularities is given by the Hilbert scheme \( X^{[n]} \) of points on \( X \) which is a fine moduli space of zero-dimensional subschemes of \( X \). The McKay correspondence can then be expressed as an equivalence of derived categories \( D(X^{[n]}) \cong D_{\mathfrak{S}_n}(X^n) \) of (\( \mathfrak{S}_n \)-equivariant) coherent sheaves; see [BKR01; Hai01].

Besides being a very interesting theoretical result, the derived McKay correspondence can be used as a computational tool for the study of vector bundles, or, more generally, sheaves and complexes thereof, on the Hilbert schemes of points on surfaces. Concretely, given a vector bundle on \( X^{[n]} \), the derived McKay correspondence \( D(X^{[n]}) \cong D_{\mathfrak{S}_n}(X^n) \) can be used to translate this bundle into a complex in \( D_{\mathfrak{S}_n}(X^n) \). Then, the homological invariants of the
vector bundle agree with those of the associated equivariant complex but the computations are often easier for the latter.

A very interesting class of vector bundles on $X^{[n]}$ is given by the tautological bundles $E^{[n]}$. They are associated to vector bundles $E$ on the surface $X$ by means of the universal family of the Hilbert scheme; see Definition 9.2.5 for details. These bundles were intensively studied for various reasons. First of all, it seems natural to consider tautological bundles if one is interested in the geometry of Hilbert schemes of points on surfaces. Furthermore, they have applications in the description of the cup product on the cohomology of the Hilbert scheme [Leh99; LS01; LS03], enumerative geometry [KST11; Ren12], and the strange duality conjecture for line bundles on moduli spaces of sheaves [Dan00; MO08]. Recently, they have also been considered as a source of examples of stable bundles in higher dimension; see [Sch10; Wan14; Wan16; Sta16].

In [Sca09a], Scala began to use the derived McKay correspondence to study tautological bundles and tensor powers thereof. In particular, he explicitly computed equivariant complexes in $D_{\mathfrak{S}_n}(X^n)$ corresponding to the tautological sheaves. This has been further exploited in [Sca09b; Sca15a; Kru14a; Kru14b; Mea15; MM15]. In the present paper, we consider an equivalence $D(X^{[n]}) \xrightarrow{\cong} D_{\mathfrak{S}_n}(X^n)$ which differs slightly from the one used in [Sca09a] and the subsequent papers. The main observation is that this considerably simplifies the description of the images of tautological bundles and their wedge powers under the equivalence as well as the proofs of these descriptions. As an application, we get new formulae for extension groups and Euler characteristics of bundles on the Hilbert scheme as well as simplified proofs of known formulae.

Let us describe the results of this paper in more detail. The main point in establishing the derived McKay correspondence $D(X^{[n]}) \cong D_{\mathfrak{S}_n}(X^n)$ is the identification, due to [Hai01], of the Hilbert scheme $X^{[n]}$ of points on $X$ with the fine moduli space of $\mathfrak{S}_n$-clusters on $X^n$. These $\mathfrak{S}_n$-clusters are, roughly speaking, scheme-theoretic generalisations of free $\mathfrak{S}_n$-orbits; see Subsection 9.2.4 for some more details. In particular, there is a universal family of $\mathfrak{S}_n$-clusters $Z \subset X^{[n]} \times X^n$ together with the projections

$$X^{[n]} \xleftarrow{p} Z \xrightarrow{q} X^n.$$ 

Then the 'usual' derived McKay correspondence, as considered in [BKR01; Sca09a; Sca09b; Sca15a; Kru14a; Kru14b; Mea15; MM15], is the equivalence of derived categories

$$\Phi := R p_! \circ q^*: D(X^{[n]}) \xrightarrow{\cong} D_{\mathfrak{S}_n}(X^n).$$

In [Sca09a], the image of a tautological bundle under the derived McKay correspondence is described by the formula

$$\Phi(E^{[n]}) \cong C_F^\bullet.$$ \hspace{1cm} (9.1)

Here, $C_F^\bullet$ is a complex of $\mathfrak{S}_n$-equivariant coherent sheaves on $X$ concentrated in degree zero with

$$C_F^0 = \bigoplus_{i=1}^n \text{pr}_i^* F$$

where $\text{pr}_i: X^n \to X$ is the projection to the $i$-th factor; see Subsection 9.2.6 for details on the higher degree terms of $C_F^\bullet$. The formula (9.1) has been used in [Sca09a; Sca09b; Sca15a;
Kru14a; Kru14b; Mea15; MM15] in order to prove many interesting consequences. However, the proofs are often computationally involved, mainly due to the higher degree terms of the complex \( C^*_F \).

The main observation exploited in this paper is that it has benefits to consider the derived McKay correspondence in the reverse direction

\[
\Psi := (\_)^{S_n} \circ q_* \circ Lp^* : D_{S_n}(X^n) \to D(X^{[n]})
\]

instead. The functor \( \Psi \) is again an equivalence, but not the inverse of \( \Phi \); see Proposition 9.2.9. The technical main result is that, if we replace \( \Phi \) by \( \Psi^{-1} \), the higher order terms of \( C^*_F \) vanish and we get a similarly simple description for the images of wedge powers of tautological bundles associated to line bundles on the surface.

**Theorem 9.1.1** (Theorem 9.3.6, Theorem 9.3.9).

1. For every coherent sheaf \( F \in \text{Coh} X \), we have \( \Psi(C_F^0) \cong F^{[n]} \).
2. For every line bundle \( L \in \text{Pic} X \) and \( 0 \leq k \leq n \), we have

\[
\Psi(W^k(L)) \cong \wedge^k L^{[n]} \quad \text{where} \quad W^k(L) = \bigoplus_{I \subseteq \{1, \ldots, n\}, |I| = k} \text{pr}_I^*(L^E_k).
\]

Here, \( \text{pr}_I : X^n \to X^k \) is the projection to the \( I \)-factors and \( W^k(L) \) carries a \( S_n \)-linearisation by permutation of the direct summands together with appropriate signs; see Definition 9.3.4 for details.

Objects of the form \( W^k(L) \) play an important role in the construction of exceptional sequences [1] and a categorical Heisenberg action [CL12; 8] on the equivariant derived category \( D_{S_n}(X^n) \). Hence, Theorem 9.1.1 can be seen as a step towards a geometric interpretation of these categorical constructions in terms of \( D(X^{[n]}) \); see Remark 9.3.12 for a few more details on this point of view.

However, we will mainly use Theorem 9.1.1 as a tool to compute homological invariants of tautological bundles and their wedge powers. We are able to give proofs of most of the known results on the cohomology and extension groups of these bundles which are much simpler than the original ones of [Sca09a; Sca09b; Kru14a]. Furthermore, we obtain new formulae such as

**Theorem 9.1.2** (Corollary 9.4.2). For \( K, L \in \text{Pic} X \) there are functorial isomorphisms

\[
\text{Ext}^*(\wedge^k K^{[n]}, \wedge^\ell L^{[n]}) \cong \bigoplus_{i = \max\{0, k + \ell - n\}} S^i \text{Ext}^*(K, L) \otimes \wedge^{k-i} H^*(K^V) \otimes \wedge^{\ell-i} H^*(L) \otimes S^{n+i-k-\ell} H^*(O_X).
\]

In Proposition 9.5.1 and Corollary 9.5.2 we observe that it can be very useful for the computation of tensor products of bundles on the Hilbert scheme to have descriptions of their images under both, \( \Phi \) and \( \Psi^{-1} \). As an application, we prove the formula

\[
\sum_{n=0}^{\infty} \chi(F^{[n]} \otimes \Lambda_1 L^{[n]}) Q^n = \frac{(1 + uQ) \chi(L)}{(1 - Q) \chi(O_X)} \cdot \sum_{p=1}^{\infty} (-1)^{p+1} \chi(F \otimes (L^{p-1}u^{p-1} + L^p u^p)) Q^p
\]
for a generating function of the Euler characteristics where $F \in \text{Coh} X$ and $L \in \text{Pic} X$; see Theorem 9.5.5 and Remark 9.5.6. Here, for a vector bundle $E$ of rank $r$ and a formal parameter $t$, we use the notational convention $\Lambda^i E := \sum_{i=0}^r (\wedge^i E)t^i$ as a sum in the Grothendieck group.

Theorem 9.1.2 is a generalisation and strengthening of the formula for the Euler bicharacteristics

$$\sum_{n=0}^{\infty} \chi(\Lambda_{-v} L[n], \Lambda_{-u} L[n])Q^n = \exp \left( \sum_{r=1}^{\infty} \chi(\Lambda_{-v^r} L, \Lambda_{-u^r} L) Q^r \right)$$

of [WZ14]; see Section 9.A for details. In loc. cit. formula (9.2) is conjectured to hold in greater generality. In Section 9.6, we give some restrictions to this conjecture showing that it does not hold if we replace the surface $X$ by a curve, neither if we replace the line bundle $L$ by a vector bundle of higher rank. In Proposition 9.6.3, we also do some further computations concerning tautological bundles on Hilbert schemes of points on curves.

Acknowledgements. The author thanks Jörg Schürmann for interesting discussions and Sönke Rollenske for comments on the text.

9.2 Preliminaries

9.2.1 General conventions

All our varieties are connected and defined over the complex numbers $\mathbb{C}$. For $M$ a variety, $D(M) := D^b(\text{Coh}(M))$ denotes the bounded derived category of coherent sheaves. We do not distinguish in the notation between a functor between abelian categories and its derived functor. For example, if $f : X \to Y$ is a morphism, we will write $f^* : D(Y) \to D(X)$ instead of $Lf^*$ for the derived pull-back.

9.2.2 Equivariant sheaves and derived categories

Let $G$ be a finite group acting on a variety $M$. We denote by $\text{Coh}_G(M)$ the abelian category of equivariant coherent sheaves and by $D_G(M) := D^b(\text{Coh}_G(M))$ its bounded derived category.

In this section, we collect some facts about equivariant categories and functors that we need later. We refer to [BKR01, Sect. 4] or [Ela14] for further details.

Let $H \subset G$ be a subgroup. The forgetful (also called restriction) functor $\text{Res}^H_G : \text{Coh}_G(M) \to \text{Coh}_H(M)$ has a both-sided adjoint, namely the induction functor $\text{Ind}^H_G : \text{Coh}_H(M) \to \text{Coh}_G(M)$. Concretely, for $E \in \text{Coh}(M)$, we have $\text{Ind}^H_G(E) = \oplus_{g \in G/H} g^* E$ equipped with a $G$-linearisation which combines the $H$-linearisation of $E$ with appropriate permutations of the direct summands.

In our case, the group $G$ will usually be the symmetric group $\mathfrak{S}_n$. We denote its non-trivial character by $a$ or $a_n$. We get an autoequivalence $\_ \otimes a : \text{Coh}_{\mathfrak{S}_n}(M) \to \text{Coh}_{\mathfrak{S}_n}(M)$ given by changing the sign of the linearisations appropriately (of course, there is also an endofunctor $\_ \otimes g : \text{Coh}_G(M) \to \text{Coh}_G(M)$ for an arbitrary representation $g$ of a finite group $G$).

Let $G$ act on a second smooth variety $N$ and let $f : M \to N$ be a $G$-equivariant morphism. Then pull-backs and push-forwards of equivariant sheaves inherit canonical linearisations so that we get functors $f^* : \text{Coh}_G(N) \to \text{Coh}_G(M)$ and, if $f$ is proper, $f_* : \text{Coh}_G(M) \to \text{Coh}_G(N)$. Furthermore, there are equivariant tensor products and homomorphism sheaves.

228
Restriction, induction, and tensor products by representations commute with equivariant pull-backs and push-forwards which means that we have the following isomorphisms of functors

\[
\text{Res}_* f^* \cong f^* \text{Res}_* \quad \text{Res}_* f_* \cong f_* \text{Res}_* \quad \text{Ind}_* f^* \cong f^* \text{Ind}_* \quad \text{Ind}_* f_* \cong f_* \text{Ind}_*
\]  

(9.3)

All the functors discussed above induce functors on the level of the derived categories. We write these induced functors in the same way as the functors between the abelian categories, e.g. we write \( f_* : \text{D}_G(M) \to \text{D}_G(N) \) instead of \( Rf_* : \text{D}_G(M) \to \text{D}_G(N) \).

For two objects \( E, F \in \text{D}_G(M) \), we denote the graded Hom-space by

\[
\text{Hom}_G^*(E, F) := \oplus_{i \in \mathbb{Z}} \text{Hom}_G^i(E, F) \quad \text{where} \quad \text{Hom}_G^i(E, F) := \text{Hom}_{\text{D}(M)}(E, F[i])
\]

We often suppress the restriction functor in the notation writing \( E := \text{Res}_* E \in \text{D}(M) \) for \( E \in \text{D}_G(M) \). The Hom-space \( \text{Hom}^i(E, F) := \text{Hom}_{\text{D}(M)}(\text{Res}_* E, \text{Res}_* F[i]) \) has a canonical \( G \)-action induced by the \( G \)-linearisations of \( E \) and \( F \) and the invariants under this action are the Hom-spaces in the equivariant category:

\[
\text{Hom}_G^*(E, F) \cong \text{Hom}^*(E, F)^G.
\]  

(9.4)

If \( G \) acts trivially on \( M \), a \( G \)-equivariant sheaf \( E \) is simply a sheaf together with a group action. Hence, we can take the invariants of \( E(U) \) for every open subset \( U \subset X \), which gives a functor \( (\_)^G : \text{Coh}_G(M) \to \text{Coh}(M) \). If \( f : M \to N \) is a morphism between varieties on which \( G \) acts trivially, we have

\[
(\_)^G f_* \cong f_*(\_)^G \quad \text{and} \quad (\_)^G f^* \cong f^*(\_)^G.
\]  

(9.5)

Let \( G \) act on a variety \( S \) and let \( \pi : S \to T \) be a \( G \)-invariant morphism of varieties. In other words, \( \pi \) is \( G \)-equivariant when we consider \( G \) acting trivially on \( T \). Then we write

\[
\pi^G := (\_)^G \circ \pi_* : \text{Coh}_G(S) \to \text{Coh}(T).
\]

Furthermore, we simply write \( \pi_*^G : \text{Coh}(T) \to \text{Coh}_G(S) \) for the functor which first equips every sheaf with the trivial \( G \)-action and then applies the equivariant pull-back.

**Lemma 9.2.1.** Let \( \pi : M \to M/G \) be the quotient of a smooth variety by a finite group. Then we have \( \pi_*^G \pi_*^G \cong \text{id} \) as endofunctors of the subcategory of perfect complexes \( \text{D}^{\text{perf}}(M/G) \subset \text{D}(M/G) \).

**Proof.** By definition of the quotient variety, we have \( \pi_*^G O_M \cong O_{M/G} \). Now, the assertion follows by the (equivariant) projection formula. \( \square \)

**Remark 9.2.2.** We need to restrict the pull-back to the category of perfect complexes since, if the quotient \( M/G \) is not smooth, \( \pi_*^G \) does not preserve the bounded derived category.

Let \( G \) act trivially on \( M \). The following is a straightforward generalisation of Frobenius reciprocity; compare [Dan01, Lem. 2.2] or [7, Sect. 3.5].

**Lemma 9.2.3.** There is an isomorphism of functors \( (\_)^G \text{Ind}_U^G \cong (\_)^U : \text{D}_U(M) \to \text{D}(M) \).

**Remark 9.2.4.** One direct consequence of the lemma is the following. Let \( E = \oplus_{i \in I} E_i \in \text{D}(X) \) be a finite direct sum. Assume \( E \) has a \( G \)-linearisation \( \lambda \) and there is a \( G \)-action on \( \mathcal{I} \) such that \( \lambda_g(E_i) = E_{g(i)} \). We say that \( \lambda \) induces the action on the index set \( \mathcal{I} \). Let \( \{i_1, \ldots, i_k\} \) be a set of representatives of the \( G \)-orbits of \( \mathcal{I} \) and set \( G_j := \text{stab}_G(i_j) \) for \( j = 1, \ldots, k \). Then

\[
E^G \cong \bigoplus_{j=1}^k E_i^G j.
\]  

229
9.2.3 Hilbert schemes of points and tautological sheaves

Throughout the text, $X$ will be a smooth quasi-projective surface. For a non-negative integer $n \in \mathbb{N}$, we denote by $X^{[n]}$ the Hilbert scheme of $n$ points on $X$. It is the fine moduli space of closed zero-dimensional subschemes of length $n$ of $X$. Hence, there is a universal family $\Xi \subset X^{[n]} \times X$ which is flat and finite of degree $n$ over $X$. The Hilbert scheme $X^{[n]}$ is smooth of dimension $2n$; see [Fog68, Thm. 2.4].

The symmetric group $S_n$ acts on the cartesian power $X^n$ by permutation of the factors. We call the quotient $X^{(n)} := X^n/S_n$ the $n$-th symmetric power of $X$. We write the points of $X^{(n)}$ as formal sums of points of $X$. More concretely, $x_1 + \cdots + x_n \in X^{(n)}$ is the point lying under the $S_n$-orbit of $(x_1, \ldots, x_n) \in X^n$. The Hilbert scheme is a resolution of the singularities of the symmetric power via the Hilbert-Chow morphism

$$\mu : X^{[n]} \to X^{(n)} , \quad [\xi] \mapsto \sum_{x \in \xi} \ell(O_{\xi,x}) : x.$$ 

which sends a zero-dimensional subscheme to its weighted support.

**Definition 9.2.5.** Let $\text{pr}_X : \Xi \to X$ and $\text{pr}_{X^{[n]}} : \Xi \to X^{[n]}$ be the projections from the universal family. We define the *tautological functor* by

$$\bigl(\bigcup\bigr)^{[n]} := \text{pr}_{X^{[n]*}} \circ \text{pr}_X^* : \mathcal{D}(X) \to \mathcal{D}(X^{[n]}).$$

Equivalently, $(\bigcup)^{[n]} \cong \mathcal{F} \mathcal{M}_{\mathcal{O}_{\Xi}}$ can be written as the Fourier–Mukai transform along the structure sheaf of the universal family. For $F \in \mathcal{D}(X)$, its image $F^{[n]} \in \mathcal{D}(X^{[n]})$ under the tautological functor is called the *tautological object* associated to $F$.

Let $E$ be a vector bundle on $X$ which we may consider as a complex concentrated in degree zero. Then, since $\text{pr}_{X^{[n]*}} : \Xi \to X^{[n]}$ is flat and finite of degree $n$, the object $E^{[n]}$ is a vector bundle (identified with a complex concentrated in degree zero) of rank $E^{[n]} = n \cdot \text{rank } E$. More generally, if $E \in \text{Coh}(X)$ is a coherent sheaf, its image $E^{[n]}$ is again concentrated in degree zero; see [Sca09b, Prop. 3]. Accordingly, we will also speak of *tautological bundles* and *tautological sheaves*.

9.2.4 Derived McKay correspondence

Let $G$ be a finite group acting on a smooth quasi-projective variety $M$. A *G-cluster* on $M$ is a zero-dimensional $G$-invariant closed subscheme $Z \subset M$ such that $\mathcal{O}(Z)$ is given by the regular representation $\mathbb{C}[G]$ of $G$. Every free $G$-orbit, viewed as a reduced subscheme, is a $G$-cluster. But there are also $G$-clusters whose support is a non-free $G$-orbit. These $G$-clusters are necessarily non-reduced. We denote by $G\text{-Hilb}(M)$ the fine moduli space of $G$-clusters. The scheme $G\text{-Hilb}$ can be reducible and we denote by $\text{Hilb}^G(M) \subset G\text{-Hilb}(M)$ the irreducible component containing all the points which correspond to free orbits. We call $\text{Hilb}^G(M)$ the *G-Hilbert scheme*. There is a morphism $\tau : \text{Hilb}^G(M) \to M/G$, called the *G-Hilbert–Chow morphism*, which sends $G$-clusters to the orbits on which they are supported. Let $Z \subset \text{Hilb}^G(M)$ be the universal family of $G$-clusters. We have a commutative diagram

$$\begin{array}{ccc}
Z & \xrightarrow{p} & M \\
\downarrow{q} & & \downarrow{\pi} \\
\text{Hilb}^G(M) & \xrightarrow{\tau} & M/G
\end{array}$$

230
where \( p \) and \( q \) are the projections and \( \pi \) is the quotient morphism.

**Theorem 9.2.6** ([BKR01]). Let \( \omega_M \) be a locally trivial \( G \)-bundle, which means that for every \( x \in M \) the stabiliser subgroup \( G_x \leq G \) acts trivially on the fibre \( \omega_M(x) \). Furthermore, assume that
\[
\dim(\text{Hilb}^G(M) \times_{M/G} \text{Hilb}^G(M)) \leq \dim M + 1
\]
where the fibre product is defined by the \( G \)-Hilbert–Chow morphism \( \tau: \text{Hilb}^G(M) \rightarrow M/G \).

Then \( \tau: \text{Hilb}^G(M) \rightarrow M/G \) is a crepant resolution and
\[
\Phi := p_*q^* : \mathcal{D}(\text{Hilb}^G(M)) \rightarrow \mathcal{D}_G(M)
\]
is an equivalence of triangulated categories.

In this paper, we consider the case that \( M = X^n \) is the cartesian power of a smooth quasi-projective surface \( X \) and \( G = \mathfrak{S}_n \) acts by permutation of the factors.

**Theorem 9.2.7** ([Hai01]). There is an isomorphism \( X^{[n]} \cong \text{Hilb}^{\mathfrak{S}_n}(X^n) \) which identifies \( \mu: X^{[n]} \rightarrow X^{(n)} \) and \( \tau: \text{Hilb}^{\mathfrak{S}_n}(X^n) \rightarrow X^{(n)} \).

In particular, we get a commutative diagram
\[
\begin{array}{ccc}
Z & \xrightarrow{p} & X^n \\
\downarrow q & & \downarrow \pi \\
X^{[n]} & \xrightarrow{\mu} & X^{(n)}
\end{array}
\]
where \( Z \subset X^{[n]} \times X^n \) is the universal family of \( \mathfrak{S}_n \)-clusters on \( X^n \).

One can easily check that \( \omega_X \) is locally trivial as a \( \mathfrak{S}_n \)-sheaf. Furthermore, by [Bri77], the Hilbert–Chow morphism \( \mu: X^{[n]} \rightarrow X^{(n)} \) is semi-small which means that
\[
\dim(X^{[n]} \times_{X^{(n)}} X^{[n]}) = \dim X^n = 2n.
\]

Hence, the assumptions of **Theorem 9.2.6** are satisfied which gives

**Corollary 9.2.8.** The functor \( \Phi = p_*q^* : \mathcal{D}(X^{[n]}) \rightarrow \mathcal{D}_{\mathfrak{S}_n}(X^n) \) is an equivalence.

**Proposition 9.2.9.** The functor \( \Psi := p^*\mathfrak{S}_n q^* : \mathcal{D}_{\mathfrak{S}_n}(X^n) \rightarrow \mathcal{D}(X^{[n]}) \) is an equivalence too.

**Proof.** The equivalence \( \Phi : \mathcal{D}(X^{[n]}) \rightarrow \mathcal{D}_{\mathfrak{S}_n}(X^n) \) is the equivariant Fourier–Mukai transform with kernel \( \mathcal{O}_Z \in \mathcal{D}_{\mathfrak{S}_n}(X^{[n]} \times X^n) \); see [Plo07; 1] for details on equivariant Fourier–Mukai transforms. In general, a Fourier–Mukai transform is an equivalence if and only if the Fourier–Mukai transform with the same kernel in the reverse direction is an equivalence (but these two equivalences are usually not inverse to each other); see [Huy06, Rem. 7.7]. In our case, the Fourier–Mukai transform with kernel \( \mathcal{O}_Z \) in the reverse direction is \( \Psi : \mathcal{D}_{\mathfrak{S}_n}(X^n) \rightarrow \mathcal{D}(X^{[n]}) \).

\( \square \)

### 9.2.5 Combinatorial notations

Whenever we write intervals, they are meant as subsets of the integers. Concretely, for \( a, b \in \mathbb{Z} \) with \( a \leq b \), we have \( [a, b] := \{a, a+1, \ldots, b\} \subset \mathbb{Z} \). Furthermore, for \( n \in \mathbb{N} \) a positive integer, we set \( [n] := [1, n] = \{1, \ldots, n\} \). For a subset, \( I \subset [n] \), we write \( \mathfrak{S}_I \leq \mathfrak{S}_n \) for the subgroup of permutations fixing every element of \( [n] \setminus I \). Clearly, \( \mathfrak{S}_I \cong \mathfrak{S}_{|I|} \). We write \( \mathfrak{a}_I \) for the alternating representation of \( \mathfrak{S}_I \).
9.2.6 Scala’s theorem

Scala [Sca09a; Sca09b] computed the image of tautological sheaves under the McKay correspondence $\Phi : D(X^{[n]}) \cong D_{\mathbb{G}_n}(X^n)$. We describe his result in this subsection.

Let $F \in \text{Coh}(X)$ be a coherent sheaf on the surface $X$. Note that the projection $\text{pr}_1 : X^n \to X$ to the first factor is $\mathbb{G}_{n-1}$-invariant, where $\mathbb{G}_{n-1} \cong \mathbb{G}_{[2,n]}$ acts by permutation of the last $n - 1$ factors of $X^n$. Hence, the pull-back $\text{pr}_1^* F$ carries a canonical $\mathbb{G}_{n-1}$-linearisation. We set

$$C^0_F := \text{Ind}_{\mathbb{G}_{n-1}}^{\mathbb{G}_n} \text{pr}_1^* F \cong \bigoplus_{i=1}^n \text{pr}_i^* F \in \text{Coh}_{\mathbb{G}_n}(X^n).$$

For $I \subset [n]$, we define the $I$-th partial diagonal as the reduced subvariety

$$\Delta_I := \{ (x_1, \ldots, x_n) \mid x_i = x_j \forall i, j \in I \} \subset X^n.$$  

We have an isomorphism $\Delta \cong X \times X^{n-|I|}$ where the first factor $X$ stands for the diagonal on the $I$-components. We denote by $\iota_I : X \times X^{n-|I|} \hookrightarrow X^n$ the embedding of the $I$-th partial diagonal and by $\text{pr}_I : X \times X^{n-|I|} \to X$ the projection to the first factor. Then $\iota_I \text{pr}_I^* F$ carries a canonical $\mathbb{G}_I \times \mathbb{G}_{|I|}$-linearisation and we set

$$F_I := \iota_I \text{pr}_I^* F \otimes a_I \in \text{Coh}_{\mathbb{G}_I \times \mathbb{G}_{|I|}}(X^n)$$

and, for $1 \leq p \leq n - 1$,

$$C^p_F := \text{Ind}_{\mathbb{G}_{p+1} \times \mathbb{G}_{n-p-1}}^{\mathbb{G}_n} F_{[p+1]} \cong \bigoplus_{I \subset [n], |I|=p+1} F_I \in \text{Coh}_{\mathbb{G}_n}(X^n).$$

For $I \subset J$, we have $\Delta_J \subset \Delta_I$. Hence, there are morphisms $F_I \to F_J$ given by restriction of local sections. Using appropriately alternating sums of these morphisms, we get $\mathbb{G}_n$-equivariant differentials $d^p : C^p_F \to C^{p+1}_F$; see [Sca09a, Rem. 2.2.2] for details. Hence we have defined a complex $C^*_F \in D_{\mathbb{G}_n}(X^n)$.

**Theorem 9.2.10** ([Sca09a; Sca09b]). For $F \in \text{Coh}(X)$, there are functorial isomorphisms $\Phi(F^{[n]}) \cong C^*_F$ in $D_{\mathbb{G}_n}(X^n)$.

**Remark 9.2.11.** Note that the definition of the $C^*_F$ still makes perfect sense as an object in $D_{\mathbb{G}_n}(X^n)$ if we replace the sheaf $F \in \text{Coh}(X)$ by a complex $F \in D(X)$. Also, for $F \in D(X)$, there are still morphisms $d^p : C^p_F \to C^{p+1}_F$ in $D_{\mathbb{G}_n}(X^n)$. This allows us to define $C^*_F$, for $F \in D(X)$, as a Postnikov system; for this notion which, roughly speaking, generalises the notion of complexes from abelian to triangulated categories, see e.g. [Orl03]. With this definition of $C^*_F$, the statement of **Theorem 9.2.10** remains true for $F \in D(X)$ instead of $F \in \text{Coh}(X)$.

Another way to phrase this is that there is an isomorphism of functors

$$\Phi \circ (\_)^{[n]} \cong C^*_\_ : D(X) \to D_{\mathbb{G}_n}(X^n)$$

where $C^*_\_ = \text{FM}_K^\cdot$ is the Fourier–Mukai transform along the $\mathbb{G}_n$-equivariant complex

$$K^\cdot = (0 \to \bigoplus_{i=1}^n \mathcal{O}_{D_i} \to \bigoplus_{|I|=2} \mathcal{O}_{D_I} \to \cdots \to \mathcal{O}_{D_{[n]}} \to 0)$$

on $X \times X^n$. Here, the $D_I$ are the reduced subvarieties given by $D_I := \cap_{i \in I} D_i$ where

$$X^n \cong D_i = \{ (x, x_1, \ldots, x_n) \mid x = x_i \} \subset X \times X^n.$$  

232
9.3 Tautological bundles under the derived McKay correspondence

In this section, we proof Theorem 9.1.1. For this purpose, we introduce various families related to tautological bundles over the Hilbert scheme and discuss their geometry.

9.3.1 Various universal families and their geometry

We define \( \Xi(n, k) \subset X^{[n]} \times X^k \) as the \( k \)-fold fibre product
\[
\Xi(n, k) := \Xi \times_{X^{[n]}} \cdots \times_{X^{[n]}} \Xi.
\]
It is the reduced (see [Sca09a, Sect. 1.4]) subvariety of \( X^{[n]} \times X^k \) given by
\[
\Xi(n, k) = \{(\xi, x_1, \ldots, x_k) \mid x_i \in \xi \forall i = 1, \ldots, k\}.
\]
We denote the projections by \( e_k : \Xi(n, k) \to X^{[n]} \) and \( f_k : \Xi(n, k) \to X^k \).

**Lemma 9.3.1.** For \( E \in D(X) \) and \( k \in \mathbb{N} \), we have natural isomorphisms
\[
\wedge^k (E^{[n]}) \cong e_k^* f_k^*(E^{[2k]} \otimes a_k).
\]

**Proof.** By flat base change along the cartesian diagram
\[
\begin{array}{ccc}
\Xi(n, k) & \xrightarrow{\delta'} & \Xi^k \\
\downarrow e_k & & \downarrow pr_X^k \\
X^{[n]} & \xrightarrow{\delta} & (X^{[n]})^k
\end{array}
\]
where \( \delta \) is the diagonal embedding, we get
\[
(E^{[n]})^{\otimes k} \cong \delta^*( (E^{[n]})^{\otimes 2k} ) \cong \delta^* pr_X^{k*} (E^{[2k]} \otimes a_k) \cong e_k^* f_k^*(E^{[2k]}).
\]
The isomorphism \( (E^{[n]})^{\otimes k} \cong e_k^* f_k^*(E^{[2k]}) \) is \( S_k \)-equivariant with \( S_k \) acting on both sides by permutation of the tensor factors. Hence, taking \( S_k \)-anti-invariants on both sides gives the assertion.

In [Hai01], a morphism \( \text{Hilb}^{S_n}(X^n) \to X^{[n]} \) (which is then shown to be an isomorphism) is constructed as follows. Let \( Z \subset \text{Hilb}^{S_n}(X^n) \times X^n \) be the universal family of \( S_n \)-clusters. One shows that \( \text{id} \times pr_1(Z) \subset \text{Hilb}^{S_n} \times X \) is flat of degree \( n \) over \( \text{Hilb}^{S_n}(X^n) \) which results in the classifying morphism \( \text{Hilb}^{S_n}(X^n) \to X^{[n]} \).

A posteriori, once the inverse morphism \( X^{[n]} \to \text{Hilb}^{S_n}(X^n) \) and hence the identification \( X^{[n]} \cong \text{Hilb}^{S_n}(X^n) \) is established, one can interpret this as follows. Let \( Z \subset X^{[n]} \times X^n \) be the universal family of \( S_n \)-clusters and \( \Xi \subset X^{[n]} \times X \) be the universal family of length \( n \) subschemes of \( X \). Then \( \text{id} \times pr_1(Z) = \Xi \) and we have \( \Xi \cong Z/S_{n-1} \) with \( \text{id} \times pr_1 \) being the quotient morphism.

233
More generally, let \( k \in [n] \) and denote by \( \text{pr}_{[k]} : X^n \to X^k \) the projection to the first \( k \) factors. We consider the closed subvariety
\[
\Xi^{(n)}_{k} := \text{id} \times \text{pr}_{[k]}(Z) = \{ (\xi, x_1, \ldots, x_k) \mid \mu(\xi) \geq x_1 + \cdots + x_k \} \subset X^n \times X^k
\]
where the inequality \( \mu(\xi) \geq x_1 + \cdots + x_k \) means that every point occurs in the left-hand side with at least the same multiplicity as in the right-hand side. Note that \( \Xi^{(n)}_{1} = \Xi \). We denote the restriction of the projection \( \text{id} \times \text{pr}_{[k]} \) by \( q_k : Z \to \Xi^{(n)}_{k} \).

**Proposition 9.3.2.** The projection \( q_k : Z \to \Xi^{(n)}_{k} \) is the \( \mathfrak{S}_{[k+1,n]} \)-quotient of \( Z \).

Since \( Z \) is the universal family of \( \mathfrak{S}_n \)-clusters, the proposition follows from this

**Lemma 9.3.3.** Let \( J \subset \mathbb{C}[x_1, y_1, \ldots, x_n, y_n] \) be the vanishing ideal of a \( \mathfrak{S}_n \)-cluster on \( (\mathbb{A}^2)^n \). Then, for \( k \leq n \), the inclusion \( \mathbb{C}[x_1, y_1, \ldots, x_k, y_k] \hookrightarrow \mathbb{C}[x_1, y_1, \ldots, x_n, y_n] \) induces an isomorphism
\[
\mathbb{C}[x_1, y_1, \ldots, x_k, y_k]/(J \cap \mathbb{C}[x_1, y_1, \ldots, x_k, y_k]) \cong (\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]/J)^{\mathfrak{S}_{n-k}}.
\]

**Proof.** For \( k = 1 \), the proof can be found in [Hai99, Sect. 4] and it works the same for arbitrary \( k \). We reproduce the argument for completeness sake. For a finite set of variables \( \{x_i, y_i\}_{i \in I} \) and non-negative integers \( h, k, n \in \mathbb{N} \), we consider the \( \mathfrak{S}_I \)-invariant power sum polynomial
\[
p_{h,k}(\{x_i, y_i\}_{i \in I}) = \sum_{i \in I} x_i^h y_i^k.
\]
By definition of a \( \mathfrak{S}_n \)-cluster, \( \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]/J \) is given by the regular representation. In particular, its \( \mathfrak{S}_n \)-invariants are one-dimensional. Hence, every power sum polynomial \( p_{h,k}(x_1, y_1, \ldots, x_n, y_n) \) is congruent to a constant \( c_{h,k} \) modulo \( J \). By a theorem of Weyl, \( \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]^{\mathfrak{S}_{n-k}} \) is generated by the power sums \( p_{h,k}(x_{k+1}, y_{k+1}, \ldots, x_n, y_n) \), as a \( \mathbb{C}[x_1, y_1, \ldots, x_k, y_k] \)-algebra; see e.g. [Dal99, Thm. 1.2]. We have
\[
p_{h,k}(x_{k+1}, y_{k+1}, \ldots, x_n, y_n) = p_{h,k}(x_1, y_1, \ldots, x_n, y_n) - p_{h,k}(x_1, y_1, \ldots, x_k, y_k)
\equiv c_{h,k} - p_{h,k}(x_1, y_1, \ldots, x_k, y_k) \mod J.
\]
This shows that the inclusion
\[
\mathbb{C}[x_1, y_1, \ldots, x_k, y_k]/(J \cap \mathbb{C}[x_1, y_1, \ldots, x_k, y_k]) \to (\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]/J)^{\mathfrak{S}_{n-k}}
\]
is also surjective. \( \square \)

**9.3.2 Tautological objects under the derived McKay correspondence**

**Definition 9.3.4.** We consider the functor
\[
\mathcal{C} := \text{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \text{pr}^*_X : \text{D}(X) \to \text{D}_{\mathfrak{S}_n}(X^n).
\]
Furthermore, for \( 1 \leq k \leq n \) and \( F \in \text{D}(X) \), we set
\[
\mathcal{W}^k(F) := \text{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}(\text{pr}^*_{\mathfrak{S}_k}(L^{\otimes k} \otimes a_k)) \cong \bigoplus_{I \subset [n]} \text{pr}^*_I(L^{\otimes k}) \otimes a_I \in \text{D}_{\mathfrak{S}_n}(X^n).
\]
Note that \( \mathcal{C}(F) \cong \mathcal{C}_F^0 \cong \mathcal{W}^1(F) \). We also set \( \mathcal{W}^0(F) := \mathcal{O}_{X^n} \) (the structure sheaf equipped with the canonical linearisation).
We denote by \( e'_k : \Xi(n)_k \rightarrow X[n] \) and \( f'_k : \Xi(n)_k \rightarrow X[n] \) the projections.

**Proposition 9.3.5.** There are isomorphisms, functorial in \( F \in \mathcal{D}(X) \),
\[
\Psi(W^k(F)) \cong e'_k \otimes f'_k (F^{\otimes k} \otimes a_k).
\]

**Proof.** We consider the commutative diagram
\[
\begin{array}{ccc}
Z & \xrightarrow{p} & X^n \\
\downarrow q & & \downarrow pr[k] \\
\Xi(n)_{k} & \xrightarrow{f'_k} & X^k \\
\downarrow e'_k & & \\
X[n] & & \\
\end{array}
\]

We get
\[
\Psi(W^k(F)) \cong q^{\otimes n} p^* \text{Ind}_{\Xi_{k} \times \Xi_{n-k}}^{\Xi_{n}} \text{pr}_k^*(F^{\otimes k} \otimes a_k).
\]

Using the language of equivariant Fourier--Mukai transforms, we can write Lemma 9.3.1 and Proposition 9.3.5 as
\[
\wedge^k(F[n]) \cong \text{FM}_{\Xi(n)_k}(F^{\otimes k} \otimes a_k)^{\otimes k} \quad \text{and} \quad \Psi(W^k(F)) \cong \text{FM}_{\Xi(n)_k}(F^{\otimes k} \otimes a_k)^{\otimes k},
\]
respectively. Since \( \Xi(n)_k \subset \Xi(n, k) \) we get a morphism \( \mathcal{O}_{\Xi(n,k)} \rightarrow \mathcal{O}_{\Xi(n)_k} \) between the Fourier--Mukai kernels given by restriction of local regular functions.

**Lemma 9.3.7.** The restriction map \( \mathcal{O}_{\Xi(n,k)} \rightarrow \mathcal{O}_{\Xi(n)_k} \) induces an isomorphism on the level of anti-invariants
\[
e_{k*}(\mathcal{O}_{\Xi(n,k)} \otimes a_k)^{\otimes k} \cong e'_{k*}(\mathcal{O}_{\Xi(n)_k} \otimes a_k)^{\otimes k}.
\]

**Proof.** The universal family \( Z \subset X[n] \times X^n \) is irreducible of dimension \( 2n \); see [Hai01, Prop. 3.3.2]. Thus, the same holds for the quotient \( \Xi(n)_k \approx Z/\Xi_{n-k} \). Hence, \( \Xi(n)_k \) is an irreducible component of \( \Xi(n, k) \) as the latter is also finite over \( X[n] \) and consequently of dimension \( 2n \). Since both, \( \Xi(n)_k \) and \( \Xi(n, k) \), are reduced, it is sufficient to show that every \( \otimes_k \)-anti-invariant function \( f \in \mathcal{O}_{\Xi(n,k)} \) vanishes on the complement of \( \Xi(n)_k \).

Every point \( p \in \Xi(n,k) \setminus \Xi(n)_k \) is of the form \( p = (\xi, x_1, \ldots, x_k) \) where \( x_i = x_j \) for at least one pair \( 1 \leq i < j \leq n \). Set \( \tau = (i \ j) \in \otimes_k \). Then \( \tau(p) = p \). It follows that every \( \otimes_k \)-anti-invariant function satisfies \( f(p) = -f(p) \), hence \( f(p) = 0 \). □
Corollary 9.3.8. For $L \in \text{Pic} X$ and $0 \leq k \leq n$, we have
\[ \wedge^k (L[n]) \cong \text{FM}_{\mathcal{O}_X}^{\mathcal{E}_k} (L \otimes a_k)^{\mathcal{E}_k} \cong \epsilon^* \delta^* f_k^*(L \otimes a_k). \]

Proof. This follows from Lemma 9.3.1 together with Lemma 9.3.7. 

Theorem 9.3.9. For $L \in \text{Pic}(X)$ and $0 \leq k \leq n$, we have $\Psi(W^k(L)) \cong \wedge^k (L[n])$.

Proof. This follows from Proposition 9.3.5 together with Corollary 9.3.8. 

Remark 9.3.10. The $k = 0$ case of Theorem 9.3.9 says that $\Psi(O_X) \cong O_X$. The $k = n$ case gives $\Psi(O_X \otimes \mathfrak{a}) \cong \text{det} O_{[n]} \cong O(-B/2)$ where $B \subset X^{[n]}$ is the effective divisor parametrizing non-reduced length $n$ subschemes of $X$.

Remark 9.3.11. Using the commutative diagram (9.6), one can easily show that the functor $\Psi: \text{D}_{\mathfrak{E}_n}(X^n) \to \text{D}(X^n)$ is $O_{X^{[n]}}$-linear, which means that, for $A \in \text{D}^\text{perf}(X^{[n]})$ and $B \in \text{D}_{\mathfrak{E}_n}(X^n)$, we have functorial isomorphisms $\Psi(\pi^* A \otimes B) \cong \mu^*(A) \otimes \Psi(B)$. Hence, Theorem 9.3.6 and Theorem 9.3.9 actually give descriptions for the image of objects under the derived McKay correspondence for a much larger class than only tautological objects and wedge powers of tautological bundles. In particular, let $M \in \text{Pic} X$ be a line bundle. Then the equivariant line bundle $M^{\mathcal{E}_n} \in \text{Pic}_{\mathfrak{E}_n}(X^n)$ descends to the line bundle $M^{(n)} := \pi^* M^{\mathcal{E}_n} \in \text{Pic} X^{(n)}$. Set $D_M := \mu^* (M^{(n)})$. Then, for $L \in \text{Pic} X$ and $F \in \text{D}(X)$, we get
\[ \Psi(C(F) \otimes M^{\mathcal{E}_n}) \cong F^{[n]} \otimes D_M, \quad \Psi(W^k(L) \otimes M^{\mathcal{E}_n}) \cong \wedge^k (L[n]) \otimes D_M. \]

Remark 9.3.12. For $Y$ a smooth projective variety, the equivariant derived category $\text{D}_{\mathfrak{E}_n}(Y^n)$ has some interesting features. For example, if $\text{D}(Y)$ carries a (full, strong) exceptional collection one can construct a (full, strong) exceptional collection on $\text{D}_{\mathfrak{E}_n}(Y^n)$. The same holds for semi-orthogonal decompositions and tilting bundles; see [1, Sect. 4]. Furthermore, there is always an action of a Heisenberg algebra on the category $\text{D}_{\mathfrak{E}_n}(Y^n)$; see [CL12] and [8]. If $Y = X$ is a surface, there is the McKay equivalence $\text{D}(X^{[n]}) \cong \text{D}_{\mathfrak{E}_n}(X^n)$. Hence, abstractly, we know that the derived category $\text{D}(X^{[n]})$ carries all the above features too. However, it would be interesting to understand the constructions (of exceptional sequences, the Heisenberg action, etc.) concretely in geometric terms on the Hilbert scheme. Theorem 9.3.9 can be seen as a step into that direction as the objects $W^k(E)$ play an important role in all these constructions on $\text{D}_{\mathfrak{E}_n}(X^n)$.

Remark 9.3.13. For $n = 2$, the autoequivalence $\Phi \Psi$ of $\text{D}_{\mathfrak{E}_n}(X^n)$ can be computed as
\[ \Phi \Psi \cong M_a \text{T}^{-1} \delta_a ; \quad (9.7) \]
see [2, Sect. 4.6]. Here, $M_a \in \text{Aut}(\text{D}_{\mathfrak{S}_2}(X^2))$ is the tensor product by the non-trivial character of $\mathfrak{S}_2$ and $\text{T}^{-1} \delta_a \in \text{Aut}(\text{D}_{\mathfrak{S}_2}(X^2))$ is the twist along the push-forward along the diagonal $\delta_a: D(X) \to D_{\mathfrak{S}_2}(X^2)$ which is a spherical functor; see [6]. Concretely,
\[ \text{T}^{-1} = \operatorname{cone} \left( \text{id} \to \delta_a \delta^* (\_ \otimes \mathcal{E}_2) \right) \]
as a cone of Fourier–Mukai transforms where $\varepsilon$ is the unit of adjunction which, in this case, is given by restriction of sections to the diagonal followed by projection to the invariants. Using
this, one can compute directly that, for \( n = 2 \), we have \( \Phi \Psi(C(F)) \cong C_F^* \) which explains the difference between Theorem 9.2.10 and Theorem 9.3.6. Conversely, for \( n > 2 \), the difference between \( C_F^* \) and \( C(F) \) allows to guess how the autoequivalence \( \Phi \Psi \in \text{Aut}(D_{\mathfrak{G}_n}(X^n)) \) could look like in general. Namely, one can hope that (9.7) still holds with \( \delta \) replaced by a spherical functor \( D_{\mathfrak{G}_{n-2}}(X \times X^{n-2}) \rightarrow D_{\mathfrak{G}_n}(X^n) \) whose image is supported on the big diagonal in \( X^n \); see [7, Sect. 5.9] for some speculation on these kind of spherical functors.

**Remark 9.3.14.** Let \( X = A \) be an abelian surface. Then there is the *generalised Kummer variety*, which is the subvariety \( K_{n-1}A \subset A^n \) parametrising length \( n \) subschemes of \( A \) whose weighted support adds up to \( 0 \in A \). It is smooth of dimension \( 2(n-1) \). We also consider the \( \mathfrak{G}_n \)-invariant subvariety

\[
N_{n-1}A = \{(a_1, \ldots, a_n) \mid a_1 + \cdots + a_n = 0\} \subset A^n.
\]

We denote the embeddings by \( i: K_{n-1}A \hookrightarrow A^n \) and \( j: N_{n-1}A \hookrightarrow A^n \), respectively. The equivalence \( \Psi: D_{\mathfrak{G}_n}(A^n) \cong D(A^n) \) restricts to a functor \( \Psi_K: D_{\mathfrak{G}_n}(N_{n-1}A) \cong D(K_{n-1}A) \) which is again an equivalence and satisfies

\[
\Psi_Kj^* \cong i^*\Psi. \tag{9.8}
\]

This follows from [Che02, Lem. 6.1 & Prop. 6.2], see also [Mea15, Lem. 6.2] for details of the analogous argument for \( \Phi: D(A^n) \rightarrow D_{\mathfrak{G}_n}(A^n) \) instead of \( \Psi \). Again, we get a tautological functor \( K_{n-1}: D(A) \rightarrow D(K_{n-1}A) \) by means of the Fourier–Mukai transform along the universal family of the generalised Kummer variety. It satisfies \( K_{n-1} \cong i^*(\_)[n] \). Hence, using (9.8), Theorem 9.3.6, and Theorem 9.3.9, we get an isomorphism of functors \( K_{n-1} \cong \Psi_Kj^* \mathcal{C} \) and

\[
\wedge^k(K_{n-1}(L)) \cong \Psi_K(j^*W^k(L)) \quad \text{for } L \in \text{Pic} A \text{ and } 0 \leq k \leq n - 1.
\]

**Remark 9.3.15.** Let \( F: D(X) \rightarrow D(X^{[n]} \) be the Fourier–Mukai transform along the universal ideal sheaf \( \mathcal{I}_e \) (recall that, in contrast, the tautological functor \( \_^{[n]} \) is the Fourier–Mukai transform along \( \mathcal{O}_e \)). If \( X \) is a K3 surface, \( F \) is a \( \mathbb{P}^{n-1} \)-functor which means in particular that the composition with its right-adjoint is given by

\[
F^RF \cong \text{id}_{D(X)} \oplus [-2] \oplus [-4] \oplus \cdots \oplus [-2(n-1)];
\]

see [Add16] where this result is proved using incidence subschemes of the products \( X^{[n]} \times X^{[n+1]} \). Shorter proofs using the derived McKay correspondence and Scala’s result Theorem 9.2.10 were given in [MM15; Mea15]. These proofs can be simplified further using Theorem 9.3.6 instead of Theorem 9.2.10. Furthermore, Theorem 9.3.6 explains the occurrence of the ‘truncated universal ideal functors’ of [1, Sect. 5].

### 9.4 Extension groups

Using the results from the previous section, we can derive various formula for the cohomologies and extension groups of tautological objects. For this we need the following general formula for the graded Hom-spaces in the equivariant derived category \( D_{\mathfrak{G}_n}(X^n) \) between the objects given by the constructions of Definition 9.3.4.
Proposition 9.4.1. Let $E, F \in D(X)$ and $0 \leq e, f \leq n$. Then

$$\text{Hom}^\ast_\mathbb{C}(W^e(E), W^f(F))$$

\[ \cong \bigoplus_{i=\max\{0, e+f-n\}}^{\min\{e, f\}} S^i \text{Hom}^\ast(E, F) \otimes \wedge^{-i} H^\ast(E^\vee) \otimes \wedge^{-i} H^\ast(F) \otimes S^{n+i-e-f} H^\ast(O_X). \]

Proof. The proof is an application of Remark 9.2.4 together with (9.4). Recall that, for the underlying non-equivariant objects, we have

$$W^e(E) \cong \bigoplus_{I \subset [n], |I| = e} \text{pr}_I^* E^\boxtimes;$$

see Definition 9.3.4. Hence,

$$\text{Hom}^\ast(W^e(E), W^f(F)) \cong \bigoplus_{(I, J) \in I} \text{Hom}^\ast(\text{pr}_I^* E^\boxtimes, \text{pr}_J^* F^\boxtimes)$$

where $I = \{(I, J) \mid I, J \subset [n], |I| = e, |J| = f\}$. The linearisations of $W^e(E)$ and $W^f(F)$ induce on $I$ the action $\sigma \cdot (I, J) = (\sigma(I), \sigma(J))$. A set of representatives of the orbits under this action is given by the pairs of the form

$$P_i := ([e], [i] \cup [e+1, e+f-i]) \quad \text{for } i \in [\max\{0, e+f-n\}, \min\{e, f\}].$$

The stabiliser of $P_i$ is $G_i := G \times \mathcal{G}_{e-i} \times \mathcal{G}_{f-i} \times \mathcal{G}_{n-e-f+i} \leq G_n$. By the Künneth formula, we see that $\text{Hom}^\ast(\text{pr}_{[e]}^* E^\boxtimes, \text{pr}_{[i] \cup [e+1, e+f-i]}^* F^\boxtimes)$, as a $G_i$-representation, is given by

$$\text{Hom}^\ast(\text{pr}_{[e]}^* (E^\boxtimes \otimes a_e), \text{pr}_{[i] \cup [e+1, e+f-i]}^* (F^\boxtimes \otimes a_f))$$

\[ \cong \text{Hom}^\ast(E, F) \otimes (H^\ast(E^\vee) \otimes a_{e-i}) \otimes (H^\ast(F) \otimes a_{f-i}) \otimes H^\ast(O_X)^{\otimes n+i-e-f}. \]

Hence, its $G_i$-invariants are given by

$$S^i \text{Hom}^\ast(E, F) \otimes \wedge^{-i} H^\ast(E^\vee) \otimes \wedge^{-i} H^\ast(F) \otimes S^{n+i-e-f} H^\ast(O_X)$$

and Remark 9.2.4 together with (9.4) give the result. \(\square\)

Corollary 9.4.2. For $E, F \in D(X)$, $K, L \in \text{Pic}(X)$ and $k, \ell \in [n]$, we have natural isomorphisms

$$H^\ast(F^{[n]}) \cong H^\ast(F) \otimes S^{n-1} H^\ast(O_X), \quad (9.9)$$

$$H^\ast((E^{[n]})^\vee) \cong H^\ast(E^\vee) \otimes S^{n-1} H^\ast(O_X), \quad (9.10)$$

$$\text{Hom}^\ast(E^{[n]}, F^{[n]}) \cong \text{Hom}^\ast(E, F) \otimes S^{n-1} H^\ast(O_X)$$

$$\oplus H^\ast(E^\vee) \otimes H^\ast(F) \otimes S^{n-2} H^\ast(O_X), \quad (9.11)$$

$$H^\ast(\wedge^k L^{[n]}) \cong \wedge^k H^\ast(L) \otimes S^{n-k} H^\ast(O_X), \quad (9.12)$$

$$\text{Hom}^\ast(E^{[n]}, \wedge^k L^{[n]}) \cong \text{Hom}^\ast(E, L) \otimes \wedge^{k-1} H^\ast(L) \otimes S^{n-k} H^\ast(O_X)$$

$$\oplus H^\ast(E^\vee) \otimes \wedge^k H^\ast(L) \otimes S^{n-k-1} H^\ast(O_X). \quad (9.13)$$

$$\text{Hom}^\ast(\wedge^k L^{[n]}, F^{[n]}) \cong \text{Hom}^\ast(L, F) \otimes \wedge^{k-1} H^\ast(L^\vee) \otimes S^{n-k} H^\ast(O_X)$$

$$\oplus \wedge^k H^\ast(L^\vee) \otimes H^\ast(F) \otimes S^{n-k-1} H^\ast(O_X), \quad (9.14)$$

238
\[
\text{Hom}^*(\wedge^k K^{[n]} \otimes \wedge^\ell L^{[n]}) \cong \bigoplus_{i=\max\{0, k+\ell-n\}} S^i \text{Hom}^*(K, L) \otimes \wedge^{k-i} H^*(K^\vee) \otimes \wedge^{\ell-i} H^*(L) \otimes S^{n+i-k-\ell} H^*(\mathcal{O}_X). \tag{9.15}
\]

**Proof.** Since \(\Psi : D_{\mathcal{O}_n}(X^n) \to D(X^{[n]})\) is an equivalence, we have

\[
\text{Hom}^*(\Psi(E), \Psi(F)) \cong \text{Hom}^*_E(E, F) \quad \text{for every } E, F \in D_{\mathcal{O}_n}(X^n).
\]

Using this, all the formulae follow from Theorem 9.3.6, Theorem 9.3.9, and Proposition 9.4.1. Concretely, to obtain the formulae as special cases of Proposition 9.4.1 we have to set

for (9.9): \(E = \mathcal{O}_X, e = 0, f = 1\), for (9.10): \(F = \mathcal{O}_X, e = 1, f = 0\), for (9.11): \(e = 1, f = 1\), for (9.12): \(E = \mathcal{O}_X, F = L, e = 0, f = k\), for (9.13): \(F = L, e = 1, f = k\), for (9.14): \(E = L, e = k, f = 1\), for (9.15): \(E = L, F = K, e = \ell, f = k\). \(\Box\)

**Remark 9.4.3.** Using Remark 9.3.11, one can easily generalise Corollary 9.4.2 to formulae involving tensor products by the natural line bundles \(\mathcal{D}_M\). For example, (9.13) becomes

\[
\text{Hom}^*(E^{[n]} \otimes \mathcal{D}_M, \wedge^k (L^{[n]} \otimes \mathcal{D}_N)) = \text{Hom}^*(E \otimes M, \wedge^k (L \otimes N) \otimes \wedge^{n-k} \text{Hom}^*(M, N) \oplus \text{Hom}^*(E \otimes M, N) \otimes \wedge^k \text{Hom}^*(M, L \otimes N) \otimes S^{n-k-1} \text{Hom}^*(M, N).
\]

**Remark 9.4.4.** Using Remark 9.3.14, one can derive formula, similar to those of Corollary 9.4.2, for the cohomology and extension groups of bundles on the generalised Kummer varieties; compare [Mea15, Thm. 6.9].

Formula (9.9) has been proved in different levels of generality in [Dan01; Sca09a; Sca09b]. The formulae (9.10) and (9.11) are proved in [Kru14a] and formula (9.12) is proved in [Sca09a]. To the best of the authors knowledge, (9.13), (9.14), and (9.15) are new. They generalise and strengthen the results of [WZ14] on the Euler bicharacteristic of wedge powers of tautological bundles associated to line bundles on the surface as we see in the following.

**Definition 9.4.5.** For \(k \in \mathbb{N}\) and \(\chi \in \mathbb{Z}\) we define

\[
\lambda^k \chi := \frac{1}{k!} \chi(\chi - 1) \cdots (\chi - k + 1), \quad s^k \chi := \frac{1}{k!} (\chi + k - 1)(\chi + k - 2) \cdots \chi.
\]

If \(\chi \in \mathbb{N}\), we have \(\lambda^k \chi = \binom{\chi}{k}\) and \(s^k \chi = \binom{\chi + k - 1}{k}\). It follows that, for a finite dimensional graded vector space \(V^*\), we have

\[
\chi(\wedge^k V^*) = \lambda^k \chi(V^*) \quad \text{and} \quad \chi(S^k V^*) = s^k \chi(V^*),
\]

where \(\chi(V^*) = \sum_{i \in \mathbb{Z}} (-1)^i \dim(V^i)\) is the Euler characteristic and the wedge and symmetric powers are formed in the graded sense, i.e. they take into account signs coming from the grading.

**Corollary 9.4.6.** If \(X\) is projective, we have

\[
\chi(\wedge^k K^{[n]} \otimes \wedge^\ell L^{[n]}) = \sum_{i=\max\{0, k+\ell-n\}}^{\min\{k, \ell\}} s^i \chi(K, L) \cdot \lambda^{k-i} \chi(K^\vee) \cdot \lambda^{\ell-i} \chi(L) \cdot s^{n+i-k-\ell} \chi(\mathcal{O}_X).
\]
for every $L, K \in \text{Pic} X$ and $k, \ell, n \in \mathbb{N}$ non-negative integers. In terms of generating functions this can be expressed equivalently as

$$
\sum_{n=0}^{\infty} \chi(\Lambda_n K[n], \Lambda_n L[n]) Q^n = \exp \left( \sum_{r=1}^{\infty} \chi(\Lambda_{-v^r} K, \Lambda_{-v^r} L) \frac{Q^r}{r} \right) 
$$

where for a vector bundle $E$ of rank $r$ and a formal parameter $t$, we use the convention $\Lambda t E = \sum_{i=0}^{r} (\wedge i E) t^i$ as a sum in the Grothendieck group.

**Proof.** The first formula follows directly from (9.15). The equivalence of the two formulae is shown in Section 9.A.

We will further discuss the formula for the Euler bicharacteristic in Section 9.6.

### 9.5 Tensor products and their Euler characteristic

#### 9.5.1 Invariants of tensor products under the McKay correspondence

For the computation of extension groups of sheaves or objects $F \in \mathcal{D}(X[n])$ on the Hilbert scheme, we can either use a description of $\Phi(F)$ or one of $\Psi^{-1}(F)$ and we have seen that the latter is often more convenient. For the computation of the cohomology of tensor products, however, it can be useful to have both descriptions at the same time due to the following

**Proposition 9.5.1.** For $E \in \mathcal{D}(X[n])$ and $F \in \mathcal{D}_{S_n}(X^n)$ there are functorial isomorphisms

$$
\mu_*(E \otimes \Psi(F)) \cong \pi_*^{S_n}(\Phi(E) \otimes F).
$$

**Proof.** This follows from the commutativity of the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{p} & X^n \\
| \downarrow q & & \downarrow \pi \\
X[n] & \xrightarrow{\mu} & X^{(n)}
\end{array}
$$

together with the projection formula:

$$
\mu_*(E \otimes \Psi(F)) \cong \mu_*(E \otimes q^{S_n} p^* F) \cong \mu_* q^{S_n} (q^* E \otimes p^* F) \cong \pi_*^{S_n} p_*(q^* E \otimes p^* F) \cong \pi_*^{S_n} (p_* q^* E \otimes F) \cong \pi_*^{S_n} (\Phi(E) \otimes F).
$$

**Corollary 9.5.2.** $E \in \mathcal{D}(X[n])$ and $F \in \mathcal{D}_{S_n}(X^n)$ there are functorial isomorphisms

$$
H^*(X[n], E \otimes \Psi(F)) \cong H(X^n, \pi_*^{S_n}(\Phi(E) \otimes F)) \cong H_{S_n}^*(X^n, \Phi(E) \otimes F).
$$

where $H_{S_n}^*(X^n, \Phi(E) \otimes F)$ denotes the equivariant cohomology, i.e.

$$
H_{S_n}^*(X^n, \Phi(E) \otimes F) \cong \text{Hom}_{S_n}^*(\mathcal{O}_{X^n}, \Phi(E) \otimes F) \cong H^*(X^n, \text{Res}(\Phi(E) \otimes F))^{S_n}.
$$
9.5.2 Euler characteristic of tensor products of tautological bundles

In addition to the description of $\Phi(F[n])$ of [Sca09a; Sca09b] for $F \in D(X)$ we now have a description of $\Psi^{-1}(\wedge^k L[n])$ for $L \in \text{Pic} X$ by Theorem 9.3.9. Hence, as an application of the observation made in the previous subsection, we can compute the Euler characteristic of objects of the form $F[n] \otimes \wedge^k L[n]$.

**Proposition 9.5.3.** For $F \in D(X)$, $L \in \text{Pic} X$, and $0 \leq k \leq n$, we have natural isomorphisms

$$\mu_*(F[n] \otimes \wedge^k L[n]) \cong \pi_*^S(C_F \otimes W^k(L))$$

with $\pi_*^S(C_F \otimes W^k(L)) = 0$ for $p > k$. Furthermore, $\pi_*^S(C_F \otimes W^k(L))$ is given by

$$p = 0 : \{ \pi_*^{S_{k-1} \times S_{n-k}}(pr_1^* F \otimes pr_1^*(L^{\otimes k} \otimes a_k))$$

$$\oplus \pi_*^{S_k \times S_{n-k-1}}(pr_{k-1}^* F \otimes pr_{k-1}^*(L^{\otimes k} \otimes a_k))$$

$$p \in [k-1] : \{ \pi_*^{S_{p+1} \times S_{k-p-1} \times S_{n-k}}(F[p+1] \otimes pr_{k}^*(L^{\otimes k} \otimes a_k))$$

$$\oplus \pi_*^{S_k \times S_{k-p} \times S_{n-k-1}}(F[p] \otimes pr_{k}^*(L^{\otimes k} \otimes a_k))$$

$$p = k : \pi_*^{S_k \times S_{n-k-1}}(F[k] \otimes pr_{k}^*(L^{\otimes k} \otimes a_k)).$$

**Proof.** The assertion $\mu_*(F[n] \otimes \wedge^k L[n]) \cong \pi_*^S(C_F \otimes W^k(L))$ follows from Proposition 9.5.1 together with Theorem 9.2.10 and Theorem 9.3.9.

The rest is again an application of Remark 9.2.4: We have

$$C_F \otimes W^k(L) \cong \bigoplus_{(I,J) \in \mathcal{I}} F_I \otimes pr_J^*(L^{\otimes k} \otimes a_k), \quad \mathcal{I} = \{(I,J) \mid I, J \subset [n], |I| = p + 1, |J| = k\}.$$

The $S_n$-action on $C_F \otimes W^k(L)$ induces the action $\sigma \cdot (I,J) = (\sigma(I), \sigma(J))$ on the index set $\mathcal{I}$. Hence, the stabiliser subgroups are given by

$$S_{(I,J)} = S_{I \cap J} \times S_{I \setminus J} \times S_{J \setminus I} \times S_{[n] \setminus (I \cup J)}.$$

Every transposition in $S_{I \setminus J}$ acts by $-1$ on $F_I \otimes pr_J^*(L^{\otimes k} \otimes a_k)$. Hence, the $S_{I \setminus J}$-invariants of $F_I \otimes pr_J^*(L^{\otimes k} \otimes a_k)$ vanish for $|I \setminus J| \geq 2$. $\square$

**Remark 9.5.4.** For $k = 1$, the statement of Proposition 9.5.3 remains valid if we replace the line bundle $L$ by an arbitrary object $E \in D(X)$; compare Theorem 9.3.6. Hence, we recover [Sca09a, Thm. 3.2.2] and [Sca09b, Thm. 32].

**Theorem 9.5.5.** Let $F \in D(X)$, $L \in \text{Pic} X$, and $0 \leq k \leq n$. If $X$ is projective,

$$\chi(F[n] \otimes \wedge^k L[n]) =$$

$$s^{n-k-1}\chi(O_X) \cdot \left( \sum_{p=0}^{k} (-1)^p \chi(F \otimes L^{\otimes p}) \cdot \chi^{k-p}(L) \right)$$

$$- s^{n-k}\chi(O_X) \cdot \left( \sum_{p=1}^{k} (-1)^p \chi(F \otimes L^{\otimes p}) \cdot \chi^{k-p}(L) \right).$$

241
Proof. By Proposition 9.5.3, we have \( H^* (F^{[n]} \otimes \wedge^k L^{[n]}) \cong H^* (\pi_* \mathcal{E}_n(C_F \otimes \mathcal{W}^k(L))) \). The assertion follows now from a straightforward computation of the Euler characteristics of the terms of \( \pi_* \mathcal{E}_n(C_F \otimes \mathcal{W}^k(L)) \). For example, for \( p \in [k-1] \), the Künneth formula gives

\[
H^* \left( \pi_* \mathcal{E}_p \otimes \mathcal{E}_{k-p-1} \otimes \mathcal{E}_{n-k} \left( F_{[p+1]} \otimes \text{pr}^*_k(L^{\mathcal{E}k} \otimes a_k) \right) \right)
\cong H^* (F \otimes L^{\otimes p+1}) \otimes \Lambda^{k-p-1} H^* (L) \otimes S^{n-k} H^* (\mathcal{O}_X),
\]

hence

\[
\chi(\pi_* \mathcal{E}_p \otimes \mathcal{E}_{k-p-1} \otimes \mathcal{E}_{n-k} \left( F_{[p+1]} \otimes \text{pr}^*_k(L^{\mathcal{E}k} \otimes a_k) \right)) = \chi(F \otimes L^{\otimes p+1}) \cdot \Lambda^{k-p-1} \chi(L) \cdot S^{n-k} \chi(\mathcal{O}_X). \]

\[\square\]

Remark 9.5.6. Again, it is possible to express Theorem 9.5.5 as an equality of generating functions, thus organising the formulae for varying \( n \) and \( k \) in one equation:

\[
\sum_{n=0}^{\infty} \chi([F^{[n]}] \otimes \Lambda_u L^{[n]}) Q^n = \frac{(1 + u Q) \chi(L)}{(1 - Q) \chi(\mathcal{O}_X)} \sum_{p=1}^{\infty} (-1)^{p-1} \chi(F \otimes (L^{p-1} u^{p-1} + L^p u^p)) Q^p.
\]

Remark 9.5.7. We consider the special case that \( F = L \). There is the Schur decomposition

\[
L^{[n]} \otimes \wedge^k L^{[n]} \cong \wedge^{k+1} L^{[n]} \oplus S_{(2,1,\ldots,1)} L^{[n]};
\]

see e.g. [FH91, eq. (6.9)]. We obtain a formula for the Euler characteristic \( \chi(S_{(2,1,\ldots,1)} L^{[n]}) \) of the Schur construction on the Hilbert scheme in terms of Euler characteristics of line bundles on the surface. Indeed,

\[
\chi(S_{(2,1,\ldots,1)} L^{[n]}) = \chi(L^{[n]} \otimes \wedge^k L^{[n]}) - \chi(\wedge^{k+1} L^{[n]})
\]

and we have such formulae for both terms on the right-hand side; see Corollary 9.4.6 and Theorem 9.5.5. We can also obtain something slightly stronger, a description of \( \mu_* (S_{(2,1,\ldots,1)} L^{[n]}) \). Namely, one can check that, under the isomorphism of Proposition 9.5.3, the direct summand \( \mu_* (\wedge^{k+1} L^{[n]}) \) of \( \mu_* (L^{[n]} \otimes \wedge^k L^{[n]}) \) corresponds to \( \pi_* \mathcal{E}_k \otimes \mathcal{E}_{n-k} \left( \text{pr}^*_{[k+1]} (L^{\mathcal{E}k+1} \otimes a_{k+1}) \right) \) embedded as a direct summand of \( \pi_* \mathcal{E}_k \otimes \mathcal{E}_{n-k} \left( \text{pr}^*_{[k+1]} (L \otimes \text{pr}^*_k (L^{\mathcal{E}k} \otimes a_k)) \right) \subset \pi_* \mathcal{E}_n (C^0_L \otimes \mathcal{W}^k(L)) \).

9.6 Further remarks

Setting \( K = L \) in (9.16) recovers the formula

\[
\sum_{n=0}^{\infty} \chi(\Lambda_{-w} L^{[n]}, \Lambda_{-w} L^{[n]}) Q^n = \exp \left( \sum_{r=1}^{\infty} \chi(\Lambda_{-w} L, \Lambda_{-w} L) \frac{Q^r}{r} \right) \quad (9.17)
\]

which was shown in [WZ14]. If we replace the surface \( X \) by a quasi-projective variety \( Y \) of arbitrary dimension, one can still associate to every vector bundle \( E \) on \( Y \) a tautological bundle \( E^{[n]} \) on the Hilbert scheme \( Y^{[n]} \) of \( n \) points on \( Y \) by means of the Fourier–Mukai transform along the universal family. In loc. cit. formula (9.17) is conjectured to generalise to smooth projective varieties of arbitrary dimension (instead of the surface \( X \)) and vector bundles of arbitrary rank (instead of the line bundle \( L \)). In the following, we give some restrictions to this conjecture. Namely, we prove that it does not hold if we replace \( X \) by a smooth curve neither if we replace \( L \) by a vector bundle of higher rank. For tautological bundles \( L^{[n]} \) associated to line bundles on a smooth variety \( Y \) with \( \dim Y > 2 \), however, the conjecture still seems reasonable; see Remark 9.6.5. In this case, one can also hope that the slightly more general formula (9.16) still holds.

242
9.6.1 Tautological bundles on Hilbert schemes of points on curves

For $C$ a smooth curve, the Hilbert–Chow morphism $\mu: C^{[n]} \to C^{(n)}$ is an isomorphism. Under the identification $C^{[n]} \cong C^{(n)}$, the role of the universal family of $n$-points is played by $\Xi \cong C \times C^{(n-1)} \cong C^n/\mathfrak{S}_{n-1}$. We define the reduced subscheme $D \subset C \times C^n$ as the polygraph

$$D = \{(x, x_1, \ldots, x_n) \mid x = x_i \text{ for some } i = 1, \ldots, n\}.$$ 

It is invariant under the $\mathfrak{S}_n$-action on $C \times C^n$ given by the permutation action on $C^n$.

**Lemma 9.6.1.** There is an isomorphism $D/\mathfrak{S}_n \cong \Xi$.

*Proof.* We have $\Xi \cong C^n/\mathfrak{S}_{n-1}$. The irreducible components of $D$ are given by

$$C^n \cong D_i = \{(x, x_1, \ldots, x_n) \mid x = x_i\} \subset C \times C^n$$

for $i = 1, \ldots, n$. For $I \subset [n]$, we set $D_I := \cap_{i \in I} D_i$. Since the components intersect transversely, exactly as in the surface case, we get an $\mathfrak{S}_n$-equivariant resolution

$$0 \to \mathcal{O}_D \to \bigoplus_{i=1}^n \mathcal{O}_{D_i} \to \bigoplus_{|I|=2} \mathcal{O}_{D_I} \to \cdots \to \mathcal{O}_{D_{[n]}} \to 0 \quad (9.18)$$

of $\mathcal{O}_D$; see [Sca09a, Rem. 2.2.1] or Remark 9.2.11 for details. The linearisations of the terms of this complex are given in such a way that, for $|I| \geq 2$, every transposition $(i, j)$ with $i, j \in I$ acts by $-1$ on $\mathcal{O}_{D_I}$. Hence, the $\mathfrak{S}_n$-invariants of the higher degree terms of (9.18) vanish and, if we denote the quotient morphism by $g: D \to D/\mathfrak{S}_n$, we have $g^*_{\mathbb{S}_n} \mathcal{O}_D \cong g^*_{\mathbb{S}_n} (\oplus_{i=1}^n \mathcal{O}_{D_i})$. By Remark 9.2.4, we get

$$g^*_{\mathbb{S}_n} \mathcal{O}_D \cong g^*_{\mathbb{S}_{n-1}} \mathcal{O}_{D_1} \cong g^*_{\mathbb{S}_{n-1}} \mathcal{O}_{C^n}. \quad (9.19)$$

There is a natural bijection between the $\mathfrak{S}_n$-orbits of $D$ and the $\mathfrak{S}_{n-1}$-orbits of $C^n$. Together with (9.19), this shows that $D/\mathfrak{S}_n \cong C^n/\mathfrak{S}_{n-1} \cong \Xi$. □

**Lemma 9.6.2.** Let $F = \Xi \times_{C^{[n]}} C^n$ be the fibre product defined by the projection $\Xi \to C^{[n]}$ and the $\mathfrak{S}_n$-quotient morphism $C^n \to C^{[n]}$. Then there is an isomorphism $F \cong D$.

*Proof.* The fibre product $F$ is flat over the smooth variety $\Xi = C \times C^{(n-1)}$, hence Cohen–Macaulay. It follows that $F$ is reduced since it is generically reduced. As a subset, the fibre product $F \subset C \times C^{(n-1)} \times C^n$ is given by

$$F = \{(x, x_2 + \cdots + x_n, y_1, \ldots, y_n) \mid x + x_2 + \cdots + x_n = y_1 + \cdots + y_n\}.$$ 

Hence, the projection $C \times C^{(n-1)} \times C^n \to C \times C^n$ induces a morphism $F \to D$ which is a bijection. We get the inverse morphism by applying the universal property of the fibre product to the projection $D \to C^n$ and the $\mathfrak{S}_n$-quotient morphism $D \to \Xi$. □

In complete analogy to the surface case, we define tautological objects using the Fourier–Mukai transform along the universal family as $F^{[n]} := \text{FM}_{\mathcal{O}_D}(F) \in \mathcal{D}(C^{[n]})$ for $F \in \mathcal{D}(C)$. We also define the $\mathfrak{S}_n$-equivariant objects $C^*_F, \mathcal{C}(F), \mathcal{W}^k(F) \in \mathcal{D}_{\mathfrak{S}_n}(C^n)$ in the same way as in the surface case; see Subsection 9.2.6 and Definition 9.3.4. Let $\pi: C^n \to C^{(n)}$ denote the quotient morphism. Since the Hilbert–Chow morphism is an isomorphism in the curve case, we can interpret the functor $\pi^*_{\mathbb{S}_n}: \mathcal{D}_{\mathfrak{S}_n}(C^n) \to \mathcal{D}(C^{(n)})$ as playing the role of $\Psi: \mathcal{D}_{\mathfrak{S}_n}(X^n) \to \mathcal{D}(X^{[n]})$ from the surface case. Also, $\pi^*_{\mathbb{S}_n}: \mathcal{D}(C^{(n)}) \to \mathcal{D}_{\mathfrak{S}_n}(C^n)$ plays the role of $\Phi: \mathcal{D}(X^{[n]}) \to \mathcal{D}_{\mathfrak{S}_n}(X^n)$. However, these two functors are not equivalences in the curve case, but $\pi^*_{\mathbb{S}_n}: \mathcal{D}(C^{(n)}) \to \mathcal{D}_{\mathfrak{S}_n}(C^n)$ is still fully faithful; see Lemma 9.2.1.
**Proposition 9.6.3.** For $E, F \in D(C)$ and $L \in \text{Pic} C$, we have

\begin{align}
\pi^* F[n] &\cong C_F^* , \quad (9.20) \\
\pi_*^\mathbb{S}_n C(F) &\cong F[n] , \quad (9.21) \\
\pi_*^\mathbb{S}_n W^k(L) &\cong \wedge^k L[n] , \quad (9.22) \\
E[n] \otimes F[n] &\cong \pi_*^\mathbb{S}_n (C(E) \otimes C_F^*) , \quad (9.23) \\
\text{Hom}(E[n], F[n]) &\cong \pi_*^\mathbb{S}_n \text{Hom}(C_E^*, C(F)) \cong \pi_*^\mathbb{S}_n \text{Hom}(C(E), C_F^* \otimes a_n) , \quad (9.24) \\
F[n] \otimes \wedge^k L[n] &\cong \pi_*^\mathbb{S}_n (C_F^* \otimes W^k(L)) , \quad (9.25) \\
\text{Hom}(\wedge^k L[n], F[n]) &\cong \pi_*^\mathbb{S}_n \text{Hom}(C_E^*, W^k(L)) , \quad (9.26) \\
\text{Hom}(\wedge^k L[n], F[n]) &\cong \pi_*^\mathbb{S}_n \text{Hom}(W^k(L), C_F^* \otimes a_n) . \quad (9.27)
\end{align}

**Proof.** By the previous lemma, we have a cartesian diagram

\[
\begin{array}{ccc}
D & \longrightarrow & C^n \\
\downarrow & & \downarrow \pi \\
\Xi & \longrightarrow & C[n] = C^{(n)} \\
\downarrow & & \downarrow \\
C & & \\
\end{array}
\]

Hence, by flat base change, we get $\pi^* F[n] \cong \text{FM}_{\mathcal{O}_D}(F)$. Now, the proof of (9.20) can be done using the resolution (9.18) in the same way as in the surface case; see [Sca09a, Thm. 2.2.3] or [Sca09b, Thm. 16] or Remark 9.2.11.

For the verification of (9.21), we use the commutative diagram

\[
\begin{array}{ccc}
C^n & \stackrel{\text{pr}_1}{\longrightarrow} & C \\
\downarrow q_1 & & \downarrow \text{pr}_C \\
\Xi = C \times C^{(n-1)} & \stackrel{\text{pr}_{C[n]}}{\longrightarrow} & C[n] \\
\end{array}
\]

where $q_1$ is the $\mathbb{S}_{n-1}$-quotient morphism and imitate the proof of Proposition 9.3.5.

Similarly, the proof of (9.22) can be done in analogy to the proof of Theorem 9.3.9 as given in Subsection 9.3.2.

Formula (9.23) follows from (9.21), the equivariant projection formula, and (9.20):

\[
E[n] \otimes F[n] \cong \pi_*^\mathbb{S}_n C(E) \otimes F[n] \cong \pi_*^\mathbb{S}_n (C(E) \otimes \pi^* F[n]) \cong \pi_*^\mathbb{S}_n (C(E) \otimes C_F^*) .
\]

The verification of the first isomorphism of (9.24) is basically the same. For the second isomorphism, note that the equivariant relative canonical bundle of the quotient is given by $\omega_\pi \cong \mathcal{O}_{C^n} \otimes a$; see [1, Lem. 5.10]. Hence, by equivariant Grothendieck duality, we get

\[
\text{Hom}(E[n], F[n]) \cong \text{Hom}(\pi_*^\mathbb{S}_n C(E), F[n]) \cong \pi_*^\mathbb{S}_n \text{Hom}(C(E), \pi^! F[n]) \\
\cong \pi_*^\mathbb{S}_n \text{Hom}(C(E), C_F^* \otimes a_n) .
\]
The verifications of (9.25), (9.26), and (9.27) are analogous to those of (9.23) and (9.24) using (9.22) instead of (9.21).

Now, we can apply the global section functor to both sides of the formulae of Proposition 9.6.3 to obtain formulae for the homological invariants of tautological sheaves on $C^{[n]}$ in terms of homological invariants on the curve $C$. The formulae for the cohomologies and their Euler characteristics are exactly the same as in the surface case. The reason is that (9.21) parallels Theorem 9.3.6, (9.22) parallels Theorem 9.3.9, (9.23) parallels Remark 9.5.4, and (9.25) parallels Proposition 9.5.3. The formulae for the extension groups and their Euler (bi-)characteristics, however, differ from the surface case.

**Proposition 9.6.4.** For $E, F \in D(C)$, we have

\[
\chi(E^{[n]}, F^{[n]}) = \chi(E, F)(\sum_{p=0}^{n-1}(-1)^p \lambda^{n-1-p} \chi(O_C)) + \chi(E^\vee)\chi(F)(\sum_{p=0}^{n-2}(-1)^p \lambda^{n-2-p} \chi(O_C))
\]

Proof. For $I \subset [n]$, we denote by $F_I^+$ the $\mathcal{G}_I \times \mathcal{G}_{[n]\setminus I}$-equivariant object $\iota_I p_I^* F$ with $\mathcal{G}_I$ acting trivially (recall that $\mathcal{G}_I$ acts on $F_I$ by $a_I$; see Subsection 9.2.6). Using Remark 9.2.4, we compute the degree $p$ terms of $\pi^*_E(C(E), C_F^* \otimes a) \cong \text{Hom}(E, F)$ as

\[
p = 0 : \quad \pi^{\mathcal{E}_{n-1}}_*(\text{pr}_1^* \text{Hom}(E, F) \otimes a_{[2,n]}) \oplus \pi^{\mathcal{E}_{n-2}}_*(\text{pr}_1^* E^\vee \otimes \text{pr}_2^* F \otimes a_{[3,n]}),
\]

\[
p \in [n-2] : \quad \{ \text{pr}_1^* E^\vee \otimes F^+_I[\pi_{n-1}^+] \otimes a_{[p+2, n]} \}
\]

\[
p = n-1 : \quad \pi^{\mathcal{E}_{n-1}}_*(\text{pr}_1^* E^\vee \otimes F^+_I[\pi_{n-1}^+]).
\]

We can compute the cohomology of these terms in order to get the asserted formula for the Euler characteristic. For example, we have

\[
H^*(\pi^*_E \mathcal{E}_{n-1} \otimes \pi^*_F \mathcal{E}_{n-1} \otimes a_{[p+2, n]}) \cong \text{Hom}^*(E, F) \otimes \wedge^{n-p-1} H^*(O_C).
\]

**Remark 9.6.5.** The formula of Proposition 9.6.4 differs from the one of the surface case which, by (9.11), reads

\[
\chi(E^{[n]}, F^{[n]}) = \chi(E, F)s^{n-1} \chi(O_X) + \chi(E^\vee)\chi(F)s^{n-1} \chi(O_X).
\]

As mentioned in the introduction, taking $E = F = L \in \text{Pic} C$, this implies that [WZ14, Conj. 1], which is known to be true in the surface case, cannot hold for curves. However, in [WZ14, Sect. 6], there is some evidence given for the conjecture to hold for tautological bundles on the Hilbert scheme $Y^{[n]}$ for $Y$ smooth of dimension $\dim Y > 2$. We can add a further small piece of evidence to this as follows. We consider the case $n = 2$. Then the Hilbert square $Y^{[2]}$ is smooth for $Y$ of arbitrary dimension. Furthermore, for $\dim Y > 2$, the functor $\Psi : D^b \mathcal{G}_Y(Y^{[2]}) \to D(Y^{[2]})$ is still fully faithful (but not an equivalence any more); see [2]. Note that, for $n = 2$, we have $Z \cong \Xi \cong \Xi(\frac{3}{2})$. Using this, one can check that Theorem 9.3.6 and Theorem 9.3.9 remain valid for $\dim Y > 2$ and $n = 2$. Concretely, this means that $\Psi(O_Y) \cong O_{Y^{[2]}}$, $\Psi(C(F)) \cong F^{[2]}$ for $F \in D(X)$, and $\Psi(L^{[2]} \otimes a) \cong \det L^{[2]}$ for $L \in \text{Pic} Y$. Hence, by the fully faithfulness of $\Psi$, the formulae of Corollary 9.4.2 remain valid for $n = 2$ and $\dim Y > 2$. 245
9.6.2 Wedge powers of tautological bundles of higher rank

In [WZ14, Sect. 2.3], it is conjectured that formula (9.17) generalises from line bundles to vector bundles of arbitrary rank. The following example shows that this cannot hold, even in the surface case. Indeed, if \( \text{rank} F \) is odd, formula (9.17) with \( L \) replaced by \( F \) predicts that \( \chi(\det F^{[2]}) = \lambda^2 \chi(\det F) \). However, we have the following

**Proposition 9.6.6.** Let \( X \) be a smooth projective surface and \( F = \mathcal{O}_X^{\oplus 3} \). Then

\[
\chi_X[2](\det F^{[2]}) = \lambda^2 \chi(\mathcal{O}_X) - \chi(\Omega_X).
\]

**Proof.** By [Sca15b], we have \( \Phi(\det F^{[2]}) \cong I^3_\Delta \otimes_a \). Using the short exact sequences

\[
0 \to I^i_{\Delta+1} \to I^i_\Delta \to I^i_\Delta/I^i_{\Delta+1} \to 0,
\]

we get

\[
\chi(\det F^{[2]}) = \chi(I^3_\Delta \otimes_a) = \chi(\mathcal{O}_{X^2} \otimes_a) - \chi(\mathcal{O}_{X^2}/I^2_\Delta \otimes_a) - \chi(I^2_\Delta/I^1_\Delta \otimes_a) - \chi(I^1_\Delta/I^0_\Delta \otimes_a).
\]

where the terms on the right-hand side are the Euler characteristics of the equivariant cohomology. Since the natural action of \( S_2 \) on \( I^i_{\Delta+1}/I^i_\Delta \) is given by \( a \), the invariants \( \pi_{S_2}^*(\mathcal{O}_{X^2}/I^2_\Delta \otimes_a) \) and \( \pi_{S_2}^*(I^2_\Delta/I^1_\Delta \otimes_a) \) vanish. Accordingly, also the terms \( \chi(\mathcal{O}_{X^2}/I^2_\Delta \otimes_a) \) and \( \chi(I^2_\Delta/I^1_\Delta \otimes_a) \) vanish and we get the assertion. \( \square \)

9.A Appendix: Computations with power series

Given a power series \( F(Q) \), we denote by \( F(Q)|Q^n \) the coefficient of \( Q^n \). With this notation, the verification that the two formulae of Corollary 9.4.6 are equivalent comes down to the following

**Proposition 9.A.1.**

\[
(-1)^{k+\ell} \exp \left( \sum_{r=1}^{\infty} \chi(\Lambda_{-u^r} K, \Lambda_{-u^r} L) \frac{Q^r}{r} \right) \big|_{u^k u^\ell Q^n} = \sum_{i=\max\{0,k+\ell-n\}}^{\min\{k,\ell\}} s^i \chi(K, L) \cdot \lambda^{k-i} \chi(K^\vee) \cdot \lambda^{\ell-i} \chi(L) \cdot s^{n+i-k-\ell} \chi(\mathcal{O}_X)
\]

For the proof, we use two simple auxiliary lemmas.

**Lemma 9.A.2.**

\[
\exp \left( \sum_{r=1}^{\infty} \frac{1}{r} Q^r \right) = \frac{1}{1 - Q}.
\]

**Proof.** One way to see this is to apply the logarithm to both sides. \( \square \)

**Lemma 9.A.3.** For \( k \in \mathbb{N} \) and \( \chi \in \mathbb{C} \), we have

1. \( s^k \chi = (-1)^{k} \lambda^k (-\chi) \),
2. \( (1 + Q)^{\chi} = \lambda^k \chi \),
3. \((1 - Q)^{-\chi Q_k} = s^k \chi\).

**Proof.** The verification of (i) is a direct computation using Definition 9.4.5 of the numbers \(s^k \chi\) and \(\chi^k \chi\). Part (ii) is the binomial coefficient theorem. Part (iii) follows from (i) and (ii). □

**Proof of Proposition 9.A.1.** We have

\[
\chi(\Lambda_{-v'} K, \Lambda_{-w'} L) Q^r = \chi(K, L) (v u Q)^r - \chi(K', L)(v Q)^r - \chi(L)(u Q)^r + \chi(O_X) Q^r.
\]

Hence, by Lemma 9.A.2, we get

\[
\exp \left( \sum_{r=1}^{\infty} \chi(\Lambda_{-v'} K, \Lambda_{-w'} L) \frac{Q^r}{r} \right) = (1 - vu Q)^{-\chi(K, L)} (1 - v Q)^{\chi(K', L)} (1 - u Q)^{\chi(L)} (1 - Q)^{-\chi(O_X)}.
\]

Now, the assertion follows using Lemma 9.A.3. □

The verification of Remark 9.5.6 is very similar.

**References**


Full Bibliography


[Hub] Andrew Hubery. “Notes on the octahedral axiom.” Personal homepage ()


